Balanced Dynamics for Three Dimensional Curved Flows

by
Chungu D. Lu

Department of Atmospheric Science
Colorado State University
Fort Collins, Colorado

Colorado State University

Department of Atmospheric Science

Paper No. 522

BALANCED DYNAMICS FOR
THREE DIMENSIONAL CURVED FLOWS

by

Chungu D. Lu

Department of Atmospheric Science
Colorado State University
Fort Collins, CO 80523

Spring 1993

Atmospheric Science Paper No. 522

ABSTRACT

BALANCED DYNAMICS FOR THREE DIMENSIONAL CURVED FLOWS

Modern balanced dynamical systems, such as the nondivergent barotropic model and the quasi-geostrophic and semigeostrophic theories, have been considered as alternatives to the primitive equation system in situations where the inertia-gravitational oscillations do not shape the flows significantly. These balanced theories have provided key understandings of what is known today about atmospheric and oceanic motions. These include for example, propagation of Rossby waves, synoptic weather systems associated with barotropic and baroclinic instabilities, fronts and frontogenesis, etc. There is, however, a serious limitation in these theories: they can not describe flows with large curvature, because the centrifugal force associated with the curvature of the flow is absent in the balanced assumptions of these theories. This defect of balanced systems obviously excludes their application to a large number of flow situations, especially the circular vortices which are ubiquitous in the atmosphere and ocean. The currently existing symmetric balanced theories, such as axisymmetric balanced vortex theory and zonally symmetric theory, are indeed devised for fluid motions with large curvature. These theories, however, can only describe zero-wavenumber motions (i.e., symmetric flows). Any eddy motions superimposed on the symmetric flows are absent in a complete picture of atmospheric dynamics.

Here we present a mixed-balance theory which can be regarded as the generalization of semigeostrophic theory because it properly includes the curvature effect. This theory can simultaneously be regarded as a generalization of the balanced symmetric theory because it extends to three dimensional motion. The theory involves a combined geostrophic and gradient momentum approximation and canonical transformations by a

set of quasi-Lagrangian coordinates. Formally similar to QG and SG, this system reduces to a compact mathematical formulation: a predictive equation for potential vorticity (or its reciprocal, potential pseudodensity), and a diagnostic equation which inverts PV to obtain the balanced mass and wind fields. The new balanced system preserves all the conservation principles. The linear solution of the new balanced system about a basic state Rankine vortex reveals a class of high frequency Rossby waves, which have been confirmed by comparison with the eigensolutions of the primitive equation model. These high frequency Rossby waves could be dynamically important to the stability of a circular vortex. A combined barotropic and baroclinic instability theorem of the Charney-Stern type is also derived.

The proper Hamiltonian structure associated with the primitive equations and the mixed-balance equations is explored, and their canonical equations are obtained through the Clebsch transformations.

ACKNOWLEDGEMENTS

I would like to express my sincere appreciation to my advisor, Dr. Wayne Schubert, for his guidance and support during my graduate studies in CSU. His tremendous insight and knowledge regarding this problem have contributed invaluably to this study. Many thanks also go to the other members of my advisory committee, Drs. Dave Randall, Richard Johnson, Michael Montgomery and Gerald Taylor, for their helpful comments and suggestions.

I would also like to thank Paul Ciesielski and Rick Taft for their computing assistance. I have also benefited greatly from many discussions with Dr. Michael Montgomery and Paul Ciesielski. A special note of thanks goes to Drs. Duane Stevens and Maria Flatau, for making their linear primitive equation model available to me. I would also like to extend my gratitude to my former advisor, Dr. Steven Rutledge, for his understanding and encouragement in allowing me to pursue my interest in atmospheric dynamics. Gail Cordova deserves my thanks for her help in preparation of the manuscript.

I am deeply indebted to my wife, Chuchu, for her love, support and patience. I could not have finished this work without her.

This research was supported by the National Science Foundation under Grant ATM-9115485.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

Chapter 1

INTRODUCTION

Two of the most conspicuous features of the planet on which we live are its nearly spherical geometry and its constant rotation about its own axis. These features determine to a large extent that the motions of the fluids covering the earth's surface (i.e., the atmosphere and ocean) are inevitably highly curved and inherently endowed with vorticity. Figure 1.1 is a satellite image of the Pacific basin, which shows the characteristic flow patterns on the earth. Here we would like to point out a few circulation patterns that illustrate our point: a nearly circular flow pattern associated with a tropical cyclone at 20°N over the Eastern Pacific; a maritime extratropical cyclone centered at 45°N off the west coast of North American; and the polar vortex circulation near the south pole. All these flow patterns possess large curvature, and it is most likely that the curvature vorticity is as large as the shear vorticity in these flow systems. There is no doubt that these flow systems can be understood most easily by vorticity dynamics or, more precisely, by potential vorticity dynamics when the stratification of the fluid is taken into account, in which case an accurate PV inversion operator is needed to include the curvature vorticity. The currently existing balanced models are not general enough to deal with the fluid motions discussed above. As an introduction, in this chapter we first give a general review of the historical development of balanced dynamics, after which we set forth our research objectives and illustrate how the current study fits into the general scheme of balanced dynamics.

1.1 Historical review of balanced dynamics

Atmospheric motions are governed by a set of physical laws: Newton's law of motion, the law of fluid continuity, the law of thermodynamics and the equation of state. The set

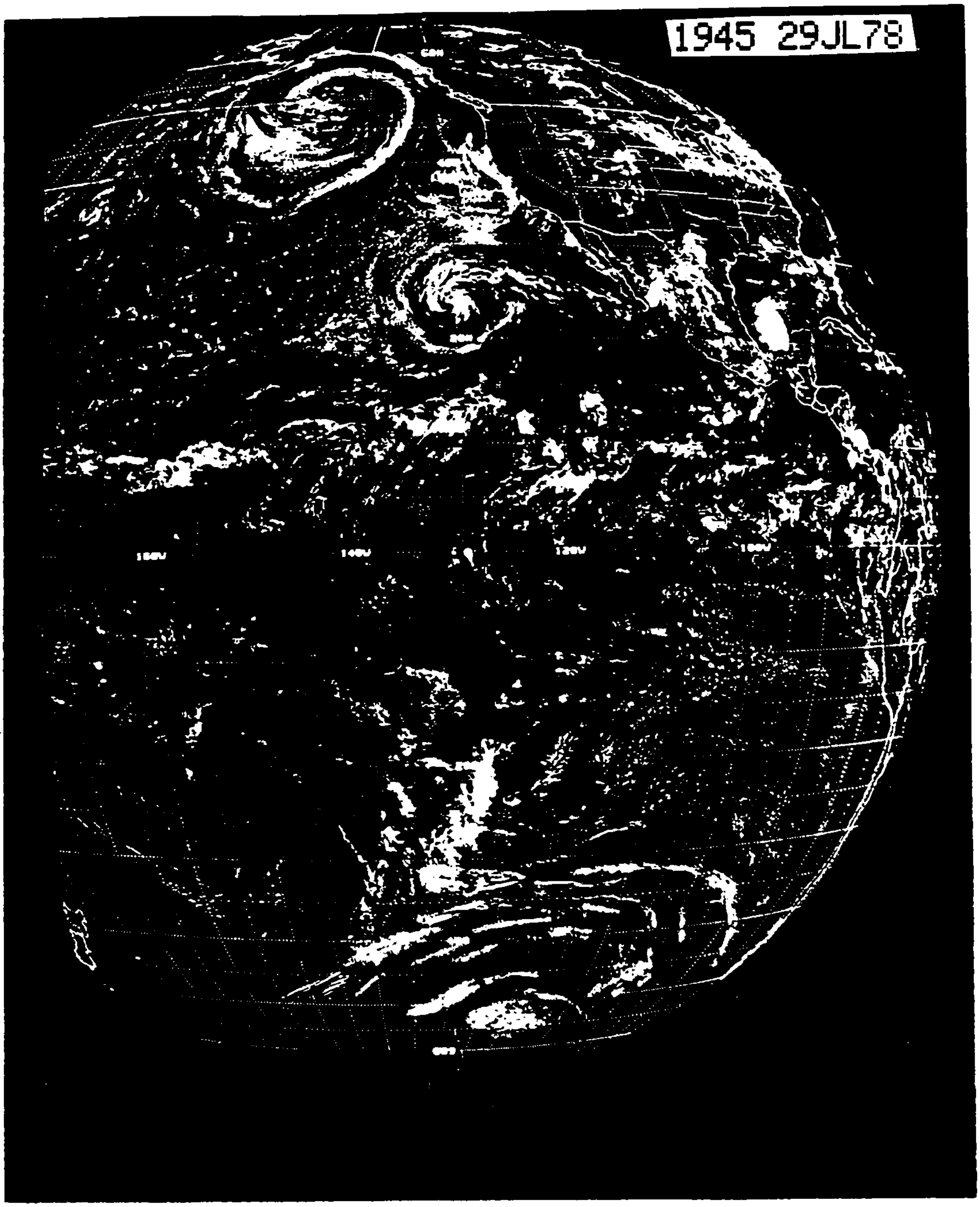


Figure 1.1: GOES VIS image of the Pacific basin at 1945 UTC on 29 July, 1978.

of partial differential equations mathematically represented these physical laws is known as a set of nonlinear wave equations, the solutions of which depict a family of wave motions in the physical world. In this sense, unlike modern physics where there was considerable debate about whether phenomena assume wave-like or particle-like properties (the well-known wave-particle duality) when the spatial scale is down to the size of molecules or atoms, the atmospheric and oceanic motions are almost certain to be engraved with wave properties. These governing equations constitute such a general dynamical system that it encompasses all the realizable motions in the atmosphere whose spectrum ranges from fast oscillating sound waves, gravitational waves to slowly varying rotational modes. The large scale motions in the atmosphere, the scale at which most weather systems manifest themselves, are mainly characterized by oscillations with low frequencies, while the acoustic and gravitational modes, though possible solutions of the original governing equations, possess insignificant amplitude at this scale (Charney, 1948). The inclusion of the fast transient modes in the study of large scale atmospheric motion is not only unnecessary, but also cumbersome because such oscillations can amplify spuriously during the integration of the governing equations even though they are very weak signals in nature, thus erroneously representing the real motions in the atmosphere (Charney, 1948; Machenhauer, 1977). The generality of such governing equations plus lack of proper initialization procedures was responsible for the failure of the earliest attempt at numerical weather prediction by Richardson in 1922.

There has been a long struggle to resolve this problem in numerical weather prediction and theoretical dynamics. Recently, there has developed the concept of "slow manifold dynamics" in which two basic approaches have been investigated to filter the unwanted frequencies (Leith, 1980). As schematically shown in Figure 1.2, the first approach is called the nonlinear normal mode initialization. The procedure is to keep the model equations unmodified but to choose the initial state to be in some sort of balance so that the fast transient noise is constantly suppressed during the model integration. This approach is preferred by numerical modelers, for it supposedly produces the most accurate simulations. The second approach, known as balanced dynamics, is to derive a set of simplified model equations from the original governing set in such a way that it only predicts the slow transient modes while preserving a set of approximated conservation principles. In the context of this approach, Lorenz (1960) has stated:

> "It is only when we use systematically imperfect equations or initial conditions that we can begin to gain further understanding of phenomena which we observe."
>
> "When the dynamic equations are to be used to further our understanding of atmospheric phenomena, it is permissible to simplify them beyond the point where they can yield acceptable weather predictions."

The simplification of the dynamic equations involves neglecting some terms in the original equations which results in the loss of some accuracy. However, the trade off for less accuracy is the simplicity of the system so that the simple, physically revealing solutions are obtainable. Furthermore, the balanced assumptions ensure the existence of the invertibility principle, which makes Ertel's law more meaningful and useful in the sense that it itself becomes the governing equation for the fundamental advective processes in the atmosphere and ocean. The coupling of these two principles forms a closed dynamical view for fluid motions governed by PV dynamics. It also provides valuable physical insights into dynamical processes in such fluid motions (Hoskins *et al.*, 1985). Therefore, this approach has been widely used among theoreticians. The current dissertation research falls into the latter category, and will henceforth focus on balanced dynamics.

Balance is literally understood as the balance of several forces in a mechanical system. Such a concept is most likely connected with the statics in which a stationary or equilibrium state is achieved under the balance of the forces. In fact, the earliest ideas of balanced dynamics in meteorology were related to the equilibrium state of air motions. The classical example of these is geostrophic balance. This balance relation is a special case of Newton's law in the sense that the particle acceleration is neglected, which results in a two-force balanced system, i.e., balance between the Coriolis force and the pressure gradient force, shown in Figure 1.3 (a). That the geostrophic relation is a pure diagnostic relation results in the complete absence of any transient wave solutions. Under this

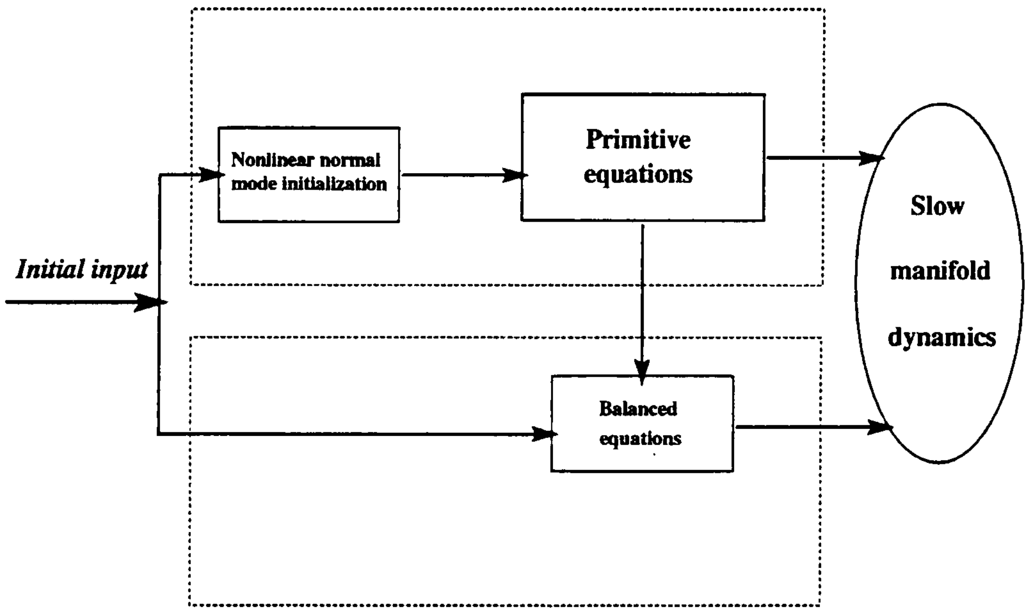


Figure 1.2: Schematic depiction of two approaches to simulating slow manifold dynamics.

balanced dynamical scenario, air steadily flows parallel to the isobars, and low pressure is always located to the left of this geostrophic flow in the northern hemisphere. This simplified dynamics historically played an important role in understading the basics of meteorological fields, and it still is the fundamental tool for synoptic weather analysis.

During the early part of this century, meteorologists began to realize that the geostrophic wind relation was not a fully valid diagnostic tool when the flow had large curvature. In fact, the velocity is often in a subgeostrophic situation when flow is curved cyclonically, and in a supergeostrophic situation when flow is curved anticyclonically. Observations show that within sharp troughs in the middle-latitude westerlies this subgeostrophy can reach as much as 50%, even though the streamlines tend to be oriented parallel to the isobars (Wallace and Hobbs, 1977, pp. 379). Under these circumstances, a third force necessarily comes into the balance relation, forming the three-force balanced system shown in Figure 1.3 (b). The balance equation involving the Coriolis force, the pressure gradient force and the centrifugal force is known as the gradient wind relation; it is commonly recognized as superior to the geostrophic wind relation when there is curvature in the fluid trajectory.

Both the geostrophic and gradient wind equations present pure diagnostic relations in which the particle acceleration is completely neglected. Therefore, all wave motions are impermissible in these balanced systems, and the flows are steady. These apparently are very crude approximations because in reality winds do change their speed and directions so that there is substantial particle acceleration. For large scale flow, however, this acceleration takes a preferential direction, i.e., the acceleration vector is quasi-horizontal. The vertical component of this acceleration is comparatively small. Neglect of the vertical acceleration results in a diagnostic relation, i.e., the hydrostatic equation. The set of primitive equations obtained by using the hydrostatic relation is the first simplified, legitimate version of the physical laws apart from the traditional approximation (Phillips, 1966). Although the quasi-static primitive equation model filters sound waves, it is still too general from the standpoint of large-scale atmospheric circulations. The further filtering process, i.e., filtering of gravity waves, has been an active subject of dynamic meteorology

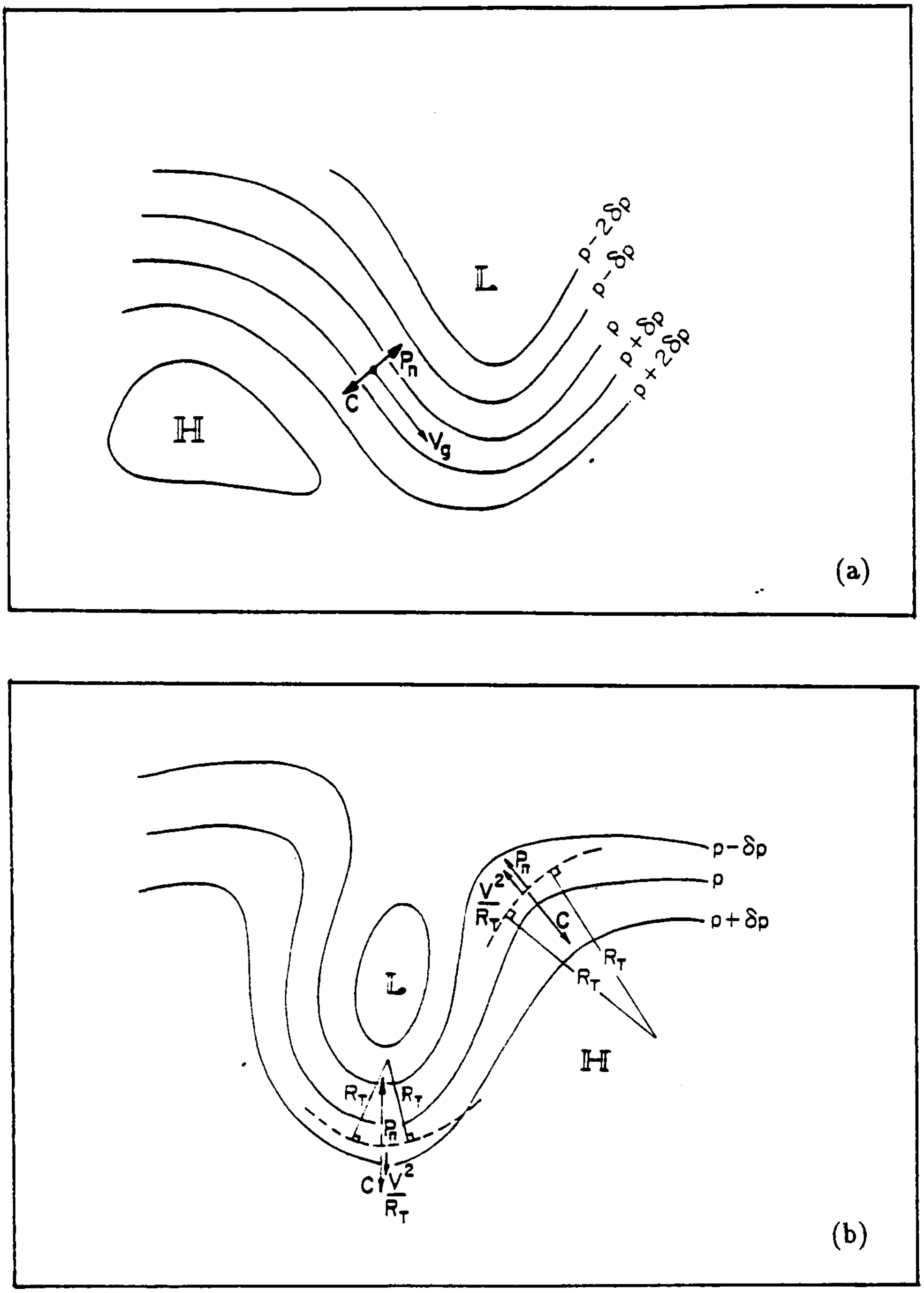


Figure 1.3: (a) Two-force balanced system. The geostrophic wind and its relation to the horizontal pressure gradient force P_n and the the Coriolis force C. (b) Three-force balanced system. Balance is achieved among the horizontal gradient force, the Coriolis force and the centrifugal force, in flow along curved trajectories in the Northern Hemisphere [adopted from Wallace and Hobbs, 1977, pp. 377-379.]

for the last several decades. We consider that the modern balanced dynamics starts here. This modern balanced dynamics is distinquished from the previous balanced systems in two ways: (1) the balance is in a more transient sense rather than in a equilibrium sense. More precisely, this means that the balanced dynamics deals with time evolution of the balanced state (e.g., geostrophic balance, or gradient balance), instead of the balanced state itself. (2) Balance takes the same meaning as filtering of gravity waves. Filtering of acoustic waves is simply not an issue here, and it is assumed that the hydrostatic approximation is valid whenever it is needed. Similar to the discussion in Magnusdottir (1989), Figure 1.4 presents a brief summary of research on this subject. On the top of the page is the primitive equation system, and various balanced models are listed below. The column on the left presents three dimensional theories, the one on the right two dimensional theories. Proceeding up the page in this figure the models become more general. They approximately follow the historical development with a somewhat interesting path in the sense that there was a big jump from the primitive equations to the first balanced model, the so-called nondivergent barotropic model, introduced by Rossby in 1939. Then the oversimplification was corrected little by little back towards the primitive equations as more generalized versions of balanced models were developed. The balanced models become more general in two ways as you go up the page. First, the earth's geometry is better represented, progressing from f-plane to β-plane to the full spherical representation. Secondly, the assumed balance becomes more general as we go from geostrophic balance to gradient wind balance to even higher order balances.

As we have mentioned previously, the nondivergent barotropic model (shown in the bottom of Figure 1.4) came as the first of its kind in filtering the gravity waves while preserving the slow rotational modes. Having realized how complex the original governing equations are, Rossby (1939) took a simplified vorticity equation that only allowed the vertical component of vorticity to be advected on a β-plane by the two dimensional, nondivergent winds. The large scale atmospheric flow can be resolved in terms of the evolution of the streamfunction which is obtained by inversion of the Laplacian operator linking it to the vorticity. By using this simple model, Rossby was able to reveal the essential physics of some sort of long waves caused by the β-effect (this was soon generalized to the sphere by Haurwitz in 1940), termed nowadays Rossby waves. More importantly, his work presented the most embryonic form of the invertibility principle.

Charney (1948) introduced the quasi-geostrophic system by using scale analysis, and later formally developed the theory in Charney and Stern (1962). The analysis of small Rossby number in these studies led to a simplification of the primitive equations in such a way that both the advected momentum and the advecting winds are replaced by their geostrophic values and the vertical advections are neglected except for the advection of the temperature field. This classical theory has been successfully applied to many midlatitude large scale phenomena. Important physical insight into the synoptic-scale cyclone waves associated with the baroclinic instability of midlatitude westerlies has been gained from this theory. Nevertheless, the severe approximations made in this system impede many applications to more general physical situations.

In order to overcome the weakness in the quasi-geostrophic system, Hoskins (1975) and Hoskins and Draghici (1977) adopted the geostrophic momentum approximation first presented by Eliassen (1948). The approximation allows one to retain a full advective operator in the Eulerian form of the equations of motion, while making the advected momentum geostrophic. A full three-dimensional vorticity equation is obtainable from the approximated system, which indicates that the dynamics of the twisting and nonlinear stretching of vorticity and the horizontal variation of static stability are all captured. Although it conceptually improves the quasi-geostrophic equations, this set is still awkward to use. The real success of semigeostrophic theory is achieved only when the geostrophic momentum approximation is accompanied by a quasi-Lagrangian coordinate transformation, which results in a compact mathematical formulation: a predictive equation for potential vorticity and a diagnostic invertibility principle to obtain the balanced wind and mass fields. A very interesting and conceptually important note is found in McWilliams and Gent (1980) and Schubert *et al.* (1989). They showed that by simultaneously using the geostrophic coordinates (with which the horizontal ageostrophic wind becomes implicit in the coordinate transformation) and the isentropic coordinate (with which the

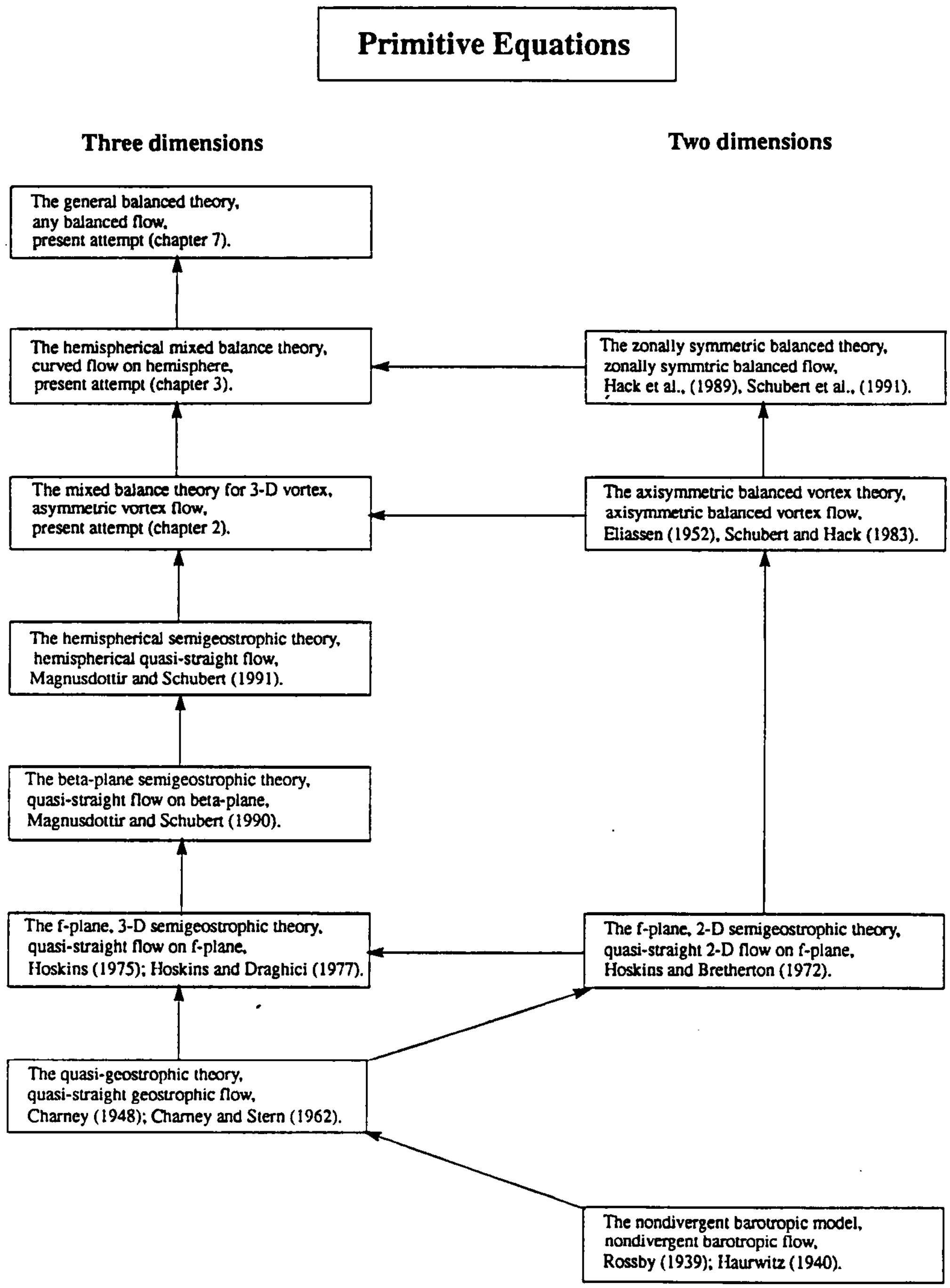


Figure 1.4: The development of balanced dynamical theories.

vertical advection vanishes for adiabatic motions), the system reduces to the form of the quasi-geostrophic set, which may be considered as the most elegant and concise version of semigeostrophic theory.

Semigeostrophic theory has had a fair measure of success in physical situations such as frontogenesis, jets, and squall lines, all of which are situations where the quasi-geostrophic theory is not applicable (Hoskins and Bretherton, 1972; Hoskins and West, 1979; Heckley and Hoskins, 1982; Montgomery and Farrell, 1991a, 1991b; Schär and Davies, 1990; Davies *et al.*, 1991; Schubert *et al.*, 1989; Hertenstein and Schubert, 1991; Fulton and Schubert, 1991).

The semigeostrophic equations were originally formulated with a constant Coriolis parameter, which severely limits applications to problems with broad spatial scale. This limitation has been overcome by several recent studies, e.g., Salmon (1985), Shutts (1989) and Magnusdottir and Schubert (1990, 1991). The former two studies derived the set of generalized semigeostrophic equations by employing Hamilton's principle, while the latter two used more conventional techniques to obtain semigeostrophic theory on the β-plane and the hemisphere.

There are some other intermediate models that are not listed in Figure 1.4. Like the semigeostrophic equations, these intermediate models contain physics between that in the primitive equations and that in the quasi-geostrophic equations. A quite complete survey and a comprehensive study of these models (including QG, GM and SG) can be found in McWilliams and Gent (1980). The solutions to different intermediate models have been calculated and comparisons of these solutions with those of a shallow water primitive equation model have been presented by Allen *et al.* (1990a, 1990b) and Barth *et al.* (1990). One class of intermediate models is the balanced equations of the Charney-Bolin type (Charney, 1955, 1962; Bolin, 1955, 1956). These simplified systems are obtained by deriving a vorticity equation and a divergence equation, and approximating the divergence equation in such a way that all the terms involving divergence of the flow field are neglected. The solution of the balanced equation models (BE) can be quite accurate in comparison with the PE solution (Moura, 1976; Allen, 1990). Lorenz (1960) has proved that these systems possess suitable energy invariants. However, due to their lack of a global conservation principle for potential vorticity, the dynamical view presented by these systems is less concise than those by QG and SG. Raymond and Jiang (1991) used the set of nonlinear balanced equations to study a mesoscale convective system. In their model, they impose a PV principle so that the entire dynamical process in a mesoscale convective system can still be interpreted in terms of IPV thinking. Recently, Allen (1991) derived a new balanced equation model based on truncations of the momentum equations (BEM) in which both the potential vorticity invariant and the energy invariant are preserved.

The term "slow manifold" used in Figure 1.2 is not intended in a rigorous sense, i.e., we do not strictly relate the balanced dynamics discussed above or developed in the present study to a true mathematical slow manifold or invariant manifold that is associated with nonlinear normal mode initialization (Leith, 1980; Lorenz, 1980). This mathematically introduced manifold has been conjectured by Leith (1980) and Lorenz (1980) as a state being completely devoid of gravity waves, i.e., the super-balanced state. There has been considerable debate about whether such a super-balanced state exists or not (Warn and Menard, 1986; Lorenz, 1986, 1987; McIntyre and Norton, 1990; 1993). Giving a general definition of balanced flow as the flow controled by PV invertibility, McIntyre and Norton argue that it is impossible to find a superinversion operator corresponding to the super-balanced state since balanced conditions and PV inversions are inherently approximate. Any vortical fluid motion described by a balanced theory will inevitably be accompanied by the spontaneous emission of inertia-gravity waves no matter how small the Froude number and Rossby number. However, the flow continually tries to adjust itself toward a better balanced state. Such spontaneous adjustment suggests the existence of the so-called "quasi-manifold" (their terminology) which is closely related to Warn's fuzzy slow manifold.

1.2 Proposed problems and research objectives

The quasi-geostrophic and semigeostrophic theories are lower order balanced models. (By lower order balance we mean that the balanced relation is a special case of a more sophisticated balanced relation, the higher order balanced relation. For example, the geostrophic balance is a special case of the gradient balance, and the gradient balance is a special case of the nonlinear balance). These models predict the future state of a two-force balanced dynamical system. For fluid motions with large curvature, e.g., those shown in Figure 1.1, the solutions from these balanced equations may differ considerably from the true solutions. Snyder *et al.* (1991) conducted a comparison study of the primitive equation model and the semigeostrophic model in the context of baroclinic waves. They found systematic discrepancies between the two model simulations. From a scale analysis, they concluded that the differences are due to the improper treatment of ageostrophic vorticity in the semigeostrophic model. While this issue needs to be further studied, we feel that the error may be partially due to the use of a lower order balanced model to simulate the highly curved flow of the baroclinic wave. As Snyder *et al.* (1991) have pointed out, within the cyclonic or anticyclonic regions associated with baroclinic waves, flows are subgeostrophic or supergeostrophic. The geostrophically balanced pressure field assumed in the semigeostrophic equations must compromise the subgeostrophic (or supergeostrophic) flows, thus resulting in less asymmetry in the geopotential field in comparison with those from PE simulations. The fact that 2-D semigeostrophic frontal simulations model (the references have been listed before) produce quite reasonable results tends to support this idea. Similar arguements can also be found in McWilliams and Gent (1980) and Snyder *et al.* (1991), who conclude that SG is a higher order approximation for two-dimensional fronts than for three-dimensional baroclinic waves.

The balanced equation models are presumed to be capable of dealing with these highly curved flows. However, it seems that these models (e.g., BE and BEM etc.) have richer physics than needed for the flows considered. This fact may explains why the balanced equation models are more complicated both physically and formally than QG and SG. Another important disadvantage associated with the balanced equation models is their lack of closed view of PV dynamics (Hoskins *et al.*, 1985). This may be closely related to the fact that these models adopt Helmholtz's decomposition of velocity field so that vorticity must be chosen as the fundamental advective quantity in accordance with the notions of streamfunction and velocity potential. In these respects, the balanced equation models seem to be designed as models computationally competitive to the primitive equation model rather than those conceptionally simpler than the primitive equation model. For our purpose here, we need to find a balanced dynamical model which is general enough to include fluid motions with large curvature, yet concise enough to preserve the same formulation as QG and SG. It is evident that such a balanced theory must assume a three-force balance, i.e., the gradient wind balance, since it is the only intermediate balanced relation between the geostrophic and the nonlinear balance equations.

There exist several theories based upon gradient balance, e.g., the axisymmetric balanced vortex theory (Eliassen, 1952; Schubert and Hack, 1983) and the zonally symmetric balanced theory (Hack *et al.*, 1989; Schubert *et al.*, 1991) as shown on the left side of Figure 1.4. These theories take the same form as the quasi-geostrophic and semigeostrophic theories, namely they reduce to one predictive equation and one invertibility principle. With one component of momentum in gradient balance, these theories can simulate the hurricane circulation (Ooyama, 1969; Schubert and Alworth, 1985), as well as the Hadley circulation (Hack *et al.*, 1989; Schubert *et al.*, 1991) fairly well. However, the deficiency associated with these theories is also quite obvious: the circulations described can only be symmetric, either axisymmetric or zonally symmetric. These theories treat the three-force balanced system as a pure equilibrium state. They cannot depict eddy motions superimposed on the symmetric circulations, and the applications must thus be confined to two dimensional flows. The three dimensional generalization of these symmetric balanced theories most likely fits into the same position as the generalization of semigeostrophic theory by inclusion flow curvature.

Craig (1991) derived a set of generalized balanced vortex equations from Hamilton's principle. Using scale analysis with the assumption that the magnitude of the radial wind is much smaller than that of the tangential wind, he was able to approximate the Lagrangian in such a form that the radial part of the wind is completely missing from the variational principle. The variations of such an approximated Hamilton's principle give rise to a set of Eulerian dynamical equations that are nearly the same as the set of Eliassen balanced vortex equations. The only difference between his set of equations and Eliassen's is the asymmetry presented in the system. Since the radial wind is absent in the approximated Lagrangian, there will be no particle acceleration in the radial direction. This radial particle acceleration seems essential to alter the axisymmetric flow. Furthermore, the vorticity vector in his system has only two components. The tangential component of the vorticity is missing due to neglect of the radial velocity. Intuitively, the asymmetric flow in a vortex is most likely to be associated with the tangential component of vorticity. Neglect of this component of vorticity seems to sacrifice the physics that is necessary to generalize Eliassen's axisymmetric vortex theory. This may indicate that Craig's model is not general enough to describe a fully three dimensional flow.

In this study, we will develop a balanced theory for fully three-dimensional, highly curved flows. Two versions of this theory are derived in separate chapters (see the outline below). The first is a theory for balanced vortices on an f-plane, and the second is the theory for planetary circulations with the centrifugal force induced by the earth's geometry. As discussed previously, this theory can be regarded as the generalization of semigeostrophic theory by using the gradient wind balance, or the generalization of symmetric balanced theory by changing the diagnostic relation for the gradient wind to a corresponding prognostic relation. The proper position of this balanced theory is shown in Figure 1.4. The upper most box is for an even higher order balanced theory yet to be discovered for more general physical situations.

The outline for the present study is as follows. In Chapter 2, we begin with the set of primitive equations in cylindrical coordinates on an f-plane. After conducting a small Rossby number analysis, we make a combined geostrophic-gradient momentum approximation in the primitive equation system. Following the formalism of semigeostrophic theory, the approximated system (which we refer to as the mixed-balance equations) is transformed to a new space constituted by a set of vortex type of coordinates. In transformed space, we are able to reduce the balanced system to two fundamental equations: one prognostic equation for potential pseudodensity and one invertibility principle. Various physical aspects are discussed during the derivations of the mixed-balance equations, and the conservation principles associated with the mixed-balance system are also derived. In Chapter 3, we generalize all the results obtained in Chapter 2 for the f-plane theory to the sphere. Therefore, the procedures are parallel to those in Chapter 2. Chapter 4 serves as a preparatory analysis for the comparisons made in Chapter 5. In this chapter, we will solve for the eigenvalues and eigenfunctions of the linear primitive equation model and of the nondivergent barotropic model. Two basic states are considered in these studies: a resting basic flow and a Rankine vortex. In Chapter 5, the mixed-balance equations are solved on both the f-plane and sphere. The eigensolutions obtained from these systems are compared with those from the primitive equations. A class of high frequency Rossby waves is identified from the eigenvalue spectrum, which may be of importance both theoretically and practically. Chapter 6 addresses the stability problems associated with the mixed-balance system. In particular, combined barotropic and baroclinic stability theorems of the Charney-Stern type are derived. The related theoretical frameworks of generalized wave-activity and Eliassen-Palm flux are discussed. In Chapter 7, we discuss balanced dynamics in the Hamiltonian mechanical framework. We first demonstrate how to use a set of Clebsch velocity potentials to transform the primitive equations to their canonical forms. The same results are obtained by combined use of Hamilton's principle and Clebsch velocity representations. If we simultaneously approximate the Lagrangian in the variational principle and Clebsch velocity potentials, the canonical equations in association with the balanced system can be obtained. This may point to a general methodology to obtain a balanced system and a general structure that a balanced system may have. Finally, in Chapter 8, summaries and conclusions are supplied.

Chapter 2

THE GENERALIZED ELIASSEN BALANCED VORTEX THEORY

In this chapter, a three dimensional balanced vortex theory will be developed. In order to give a complete discussion of this topic, we review in section 2.1 some of the basic theory of the primitive equations in cylindrical coordinates as well as the conservation properties associated with this system. Beginning in section 2.2, a mixed geostrophic-gradient balanced system will be derived and constructed. Some prior discussions such as the small Rossby number analysis, the combined geostrophic-gradient momentum approximation and the conservation theorems will be conducted in section 2.2. Section 2.3 brings in the topic of the coordinate transformation and the canonical momentum equations. In section 2.4, the potential vorticity principle (or the potential pseudodensity principle) in association with the mixed balanced system is derived. To complete the balanced model, section 2.5 addresses the practical question of how the predicted PV is inverted to give the useful dynamic and thermodynamic information, namely the question regarding the invertibility principle. Section 2.6 serves as a comparison study, in which we intend to show that for the two dimensional case, our 3-D balanced vortex theory systematically reduces to the Eliassen axisymmetric balanced vortex model (Eliassen, 1952; Shutts and Thorpe, 1978; Schubert and Hack, 1983; Schubert and Alworth, 1987).

2.1 The primitive equations

We consider a stratified fluid under the earth's gravity $\mathbf{g}$ on a rotating planet. The Euler momentum equation in vectorial form can be written

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2\mathbf{\Omega}\times\mathbf{u} + \frac{1}{\rho}\nabla p + \mathbf{g} = \mathbf{F}, \tag{2.1}$$

where $\mathbf{u} = (u, v, w)$ is the three dimensional velocity, $\boldsymbol{\Omega}$ the angular velocity of the earth, $\mathbf{F}$ the frictional or other external body forces, and the other notations are conventional in fluid mechanics and atmospheric science. To complete the dynamic model, one also needs the continuity equation, the thermodynamic equation and the equation of state. For simplicity, we will not list them here.

For meteorological applications, the traditional approximation is commonly invoked, i.e., the radial position of a fluid particle is expressed as $r = a + z$ (where a is the earth's radius and z the local vertical elevation), and the horizontal component of the Coriolis acceleration is neglected to be consistent with the total energy and angular momentum principles (Phillips, 1966). With this approximation, (2.1) becomes

$$\frac{\partial \mathbf{u}}{\partial t} + (\nabla \times \mathbf{u}) \times \mathbf{u} + f\mathbf{k} \times \mathbf{u} + \tfrac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) + \frac{1}{\rho}\nabla p + \mathbf{g} = \mathbf{F}, \tag{2.2}$$

after use of the vector identity $(\mathbf{u} \cdot \nabla)\mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u})$. The Coriolis parameter $f = 2\Omega \sin\phi$, and ϕ is the latitude. In this chapter, we will focus our attention on the f-plane problem. In other words, we will neglect the earth's geometry for the time being and, consequently, treat f as a constant in the following derivations.

We shall now choose a coordinate system to expand the governing equation (2.2). Since the theory developed in this chapter is mainly devised to study the circular type of flows in the atmosphere, we consider a set of cylindrical coordinates (r, ϕ, z) with r being the radial distance from the axis of the vortex, ϕ the azimuthal angle and z the vertical elevation. The three dimensional winds are then the radial, tangential and vertical velocities and the gradient operator expressed in this coordinate system is

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{\partial}{r\partial \phi}, \frac{\partial}{\partial z}\right). \tag{2.3}$$

The decomposition of (2.2) into three component equations in the cylindrical coordinate system results in

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + v\frac{\partial u}{r\partial \phi} + w\frac{\partial u}{\partial z} - \left(f + \frac{v}{r}\right)v + \frac{1}{\rho}\frac{\partial p}{\partial r} = F_r, \tag{2.4}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + v\frac{\partial v}{r\partial \phi} + w\frac{\partial v}{\partial z} + \left(f + \frac{v}{r}\right)u + \frac{1}{\rho}\frac{\partial p}{r\partial \phi} = F_\phi, \tag{2.5}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + v\frac{\partial w}{r\partial \phi} + w\frac{\partial w}{\partial z} + g + \frac{1}{\rho}\frac{\partial p}{\partial z} = F_z. \tag{2.6}$$

Note that unlike the fluid equations in rectangular cartesian coordinates, the nonlinear curvature vorticity appears explicitly in (2.4)–(2.5).

So far, since we have not introduced any filtering process in the governing system, (2.4)–(2.6) together with the continuity equation, thermodynamic equation and equation of state are able to resolve any mode of the motions in the atmosphere. In fact, the eigensolutions of (2.4)–(2.6) can be categorized into two linear manifolds (Leith, 1980): the slow manifold in which the flow motions are characterized by time scales larger than or comparable to one pendulum day, and the fast manifold in which the flow motions are characterized by time scales much smaller than one pendulum day. The fast class of eigenmodes first filtered from the above governing set is the acoustic waves which possess the least energy in the atmospheric motions. This filtering process is implemented by using the hydrostatic balance approximation such that

$$\frac{\partial p}{\partial z} = -\rho g, \tag{2.7}$$

which is justified for atmospheric motions in which the vertical depth scale is considerably smaller than the horizontal length scale.

Along with the introduction of this quasi-static approximation, a body of theory on vertical coordinate transformations has been developed in which the geometrical altitude need not be considered as the only choice of vertical coordinate. In fact, any piecewise monotonic function of height can be selected as the vertical coordinate through use of (2.7). A particular choice of the various vertical coordinates, such as pressure, log-pressure, sigma, pseudo-height and potential temperature, is usually more phenomenological and closely related to a particular model system. In the context of semigeostrophic models, the θ-coordinate plays a special role in, combined with the geostrophic coordinates, systematically transforming the semigeostrophic equations to an almost identical formulation to the quasi-geostrophic model (Schubert *et al.*, 1989). The discussion of the pros and cons of utilizing different variables as the vertical coordinate can be found in Kasahara

(1974). In the current study we propose to use entropy as the vertical coordinate. While retaining all the advantages of potential temperature as the vertical coordinate, the entropy coordinate provides an elegant form of the hydrostatic equation in which, unlike in the θ-coordinate (Schubert *et al.* 1989; Magnusdottir and Schubert 1990, 1991), the Exner function does not emerge and therefore we do not have to introduce this extra variable in the model equations. Moreover, the numerical computation for the invertibility principle is also simpler in the entropy coordinate (Fulton and Taft, 1991). After some elementary derivations using the specific entropy $s = c_p ln(\theta/\theta_0)$, we obtain the primitive equations expressed in the cylindrical and entropy coordinate system (r, ϕ, s),

$$\frac{Du}{Dt} - \left(f + \frac{v}{r}\right) v + \frac{\partial M}{\partial r} = F_r, \tag{2.8}$$

$$\frac{Dv}{Dt} + \left(f + \frac{v}{r}\right) u + \frac{\partial M}{r\partial \phi} = F_\phi, \tag{2.9}$$

$$\frac{\partial M}{\partial s} = T, \tag{2.10}$$

$$\frac{D\sigma}{Dt} + \sigma\left(\frac{\partial(ru)}{r\partial r} + \frac{\partial v}{r\partial\phi} + \frac{\partial \dot{s}}{\partial s}\right) = 0, \tag{2.11}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + v\frac{\partial}{r\partial\phi} + \dot{s}\frac{\partial}{\partial s} \tag{2.12}$$

is the total derivative, $M = c_p T + gz$ the Montgomery potential, $\sigma = -\partial p/\partial s$ the pseudo-density in s-space, $(u, v, \dot{s})$ the radial, tangential and vertical components of the velocity. Here we see that for adiabatic flow, the vertical advection will be implicit in the coordinate system. For diabatic flow, $\dot{s}$ is prescribed. Therefore, the thermodynamic equation no longer serves as an explicit model equation, rather it is implicit in the coordinate system. This simplification may be another advantage of using entropy as the vertical coordinate. While the frictional force and diabatic heating are either specified or given by some parameterization schemes for the pertinent physical processes, together with the equation of state, (2.8)–(2.11) form a closed system. By giving the proper initial and boundary conditions, the five equations are to be used to solve for five unknowns u, v, M, T and σ with the independent variables r, ϕ and s.

2.1.1 The conservation principles

Although the primitive system results from both the traditional and the hydrostatic approximations to the exact system, it nonetheless preserves a set of physical principles similar to those of the original system. These include the principles of angular momentum, energy, vorticity and potential vorticity. From the viewpoint of Hamiltonian mechanics, the foregoing approximations, apparently, do not destroy the intrinsic symmetries of the original Lagrangian, thus the existence of these invariants are naturally ensured by Noether's theorem (this subject will be discussed in more detail later in Chapter 7). Here we derive a set of physical laws associated with the primitive equations in a conventional manner. The purpose of the following somewhat detailed derivations is not meant to duplicate the known facts but rather is intended to set up a procedure to be used as a parallel comparison with the balanced system that will be explored in the next section.

a. Angular momentum conservation

In cylindrical coordinates the absolute angular momentum is defined as

$$m = rv + \tfrac{1}{2}fr^2. \tag{2.13}$$

On taking the material derivative of (2.13) and using (2.9), the absolute angular momentum principle is easily obtained:

$$\frac{Dm}{Dt} + \frac{\partial M}{\partial \phi} = rF_\phi. \tag{2.14}$$

For axisymmetric flow, the absolute angular momentum is a materially conserved quantity. Equation (2.14) can also be written in a flux form

$$\frac{\partial(\sigma m)}{\partial t} + \frac{\partial(\sigma rum)}{r\partial r} + \frac{\partial(\sigma vm)}{r\partial \phi} + \frac{\partial(\sigma \dot{s} m)}{\partial s} + \sigma\frac{\partial M}{\partial \phi} = \sigma r F_\phi, \tag{2.15}$$

by using the continuity equation (2.11). In an attempt to integrate this equation, we now encounter the traditional difficulty of using the isentropic coordinate system when the lower boundary is not a coordinate surface. Here we use the massless layer method to resolve this problem. The idea was originally proposed by Lorenz (1955) in defining available potential energy, and later it was adopted in many different contexts such as baroclinic instability (Bretherton, 1966; Hoskins *et al.*, 1985), surface frontogenesis (Fulton and Schubert, 1991) and generalization of the Eliassen-Palm theorem (Andrews, 1983). The idea is to assume that isentropic surfaces cross the earth's surface, continuing just under it with pressure equal to the surface pressure. At any horizontal position where two distinct isentropic surfaces run just under the earth's surface, there is no mass trapped between them so that $\sigma = 0$ there. Let us regard the bottom isentropic surface s_B as the largest value of s which remains everywhere below the earth's surface. Assuming that $\dot{s}$ vanishes at both the bottom and top boundaries, and taking the periodic condition in azimuthal direction, we then integrate (2.15) vertically from s_B to the upper isentropic surface s_T and azimuthally from ϕ to $\phi + 2\pi$. The result is:

$$\frac{\partial}{\partial t}\iint m\sigma r d\phi ds + \frac{\partial}{\partial r}\iint mu\sigma r d\phi ds = -\iint \frac{\partial M}{\partial \phi}\sigma r d\phi ds + \iint rF_\phi \sigma r d\phi ds, \qquad (2.16)$$

where it is clearly seen that the time rate of change of vertically and azimuthally integrated absolute angular momentum is due to the angular momentum flux across the radial boundaries and the generation (or dissipation) forced by the pressure and frictional torques represented by the two terms on the right-hand-side of (2.16).

b. Energy conservation

The kinetic energy principle can be obtained by adding u times (2.8) and v times (2.9). In doing so, we obtain

$$\frac{DK}{Dt} + u\frac{\partial M}{\partial r} + v\frac{\partial M}{r\partial \phi} = uF_r + vF_\phi, \qquad (2.17)$$

where $K = \frac{1}{2}(u^2 + v^2)$ is the quasi-static version of kinetic energy. Combining this result with (2.11) we obtain

$$\frac{\partial(\sigma K)}{\partial t} + \frac{\partial(\sigma ruK)}{r\partial r} + \frac{\partial(\sigma vK)}{r\partial \phi} + \frac{\partial(\sigma \dot{s} K)}{\partial s} + \sigma u\frac{\partial M}{\partial r} + \sigma v\frac{\partial M}{r\partial \phi} = \sigma(uF_r + vF_\phi). \quad (2.18)$$

After manipulation of (2.18) using the continuity and hydrostatic equations, the mass-weighted kinetic energy equation can be written as

$$\frac{\partial}{\partial t}(\sigma K) + \frac{\partial}{r\partial r}(r\sigma u(K + gz)) + \frac{\partial}{r\partial \phi}(\sigma v(K + gz)) + \frac{\partial}{\partial s}\left(\sigma \dot{s}(K + gz) - gz\frac{\partial p}{\partial t}\right)$$

$$+\sigma\alpha\omega = \sigma(uF_r + vF_\phi), \qquad (2.19)$$

where α is the specific volume and $\omega = Dp/Dt$ the vertical velocity in p-coordinate. We see from (2.19) that the local kinetic energy is changed through the total energy fluxes that cross the domain boundaries and the conversion of available potential energy. The external forces generate additional work which will dissipate (or generate) the kinetic energy.

The thermodynamic energy equation can be written in a flux form by using the mass continuity equation (2.11). This gives

$$\frac{\partial}{\partial t}(\sigma c_p T) + \frac{\partial}{r\partial r}(r\sigma u c_p T) + \frac{\partial}{r\partial \phi}(\sigma v c_p T) + \frac{\partial}{\partial s}(\sigma \dot{s} c_p T) - \sigma\alpha\omega = \sigma Q, \tag{2.20}$$

where $Q = T\dot{s}$ is the diabatic heating. The addition of (2.19) and (2.20) results in the cancellation of the conversion term $\sigma\alpha\omega$ and leads to a total energy equation

$$\begin{aligned}&\frac{\partial}{\partial t}(\sigma(K + c_p T)) + \frac{\partial}{r\partial r}(r\sigma u(K + M)) + \frac{\partial}{r\partial \phi}(\sigma v(K + M)) \\ &\quad + \frac{\partial}{\partial s}\left(\sigma\dot{s}(K + M) - gz\frac{\partial p}{\partial t}\right) = \sigma(uF_r + vF_\phi + Q).\end{aligned} \tag{2.21}$$

Again we treat the lower boundary by using the massless layer approach when the isentropes intercept by the earth's surface. Assuming the top boundary is both an isentropic and isobaric surface, assuming no topography and vanising $\dot{s}$ at the top and bottom, we perform the same integrations as we did for the angular momentum equation, which results in

$$\frac{\partial}{\partial t}\iint (K + c_p T)\sigma r d\phi ds + \frac{\partial}{\partial r}\iint (K + M)u\sigma r d\phi ds = \iint (uF_r + vF_\phi + Q)\sigma r d\phi ds. \tag{2.22}$$

Thus, in the absence of the external forcing, the vertically and azimuthaly integrated total energy is a conserved quantity.

c. The vorticity, potential vorticity and potential pseudodensity equations

To derive the vorticity equation, we first rewrite the momentum equations (2.8) and (2.9) in their rotational forms

$$\frac{\partial u}{\partial t} - \zeta v + \dot{s}\frac{\partial u}{\partial s} + \frac{\partial}{\partial r}[M + \tfrac{1}{2}(u^2 + v^2)] = F_r, \tag{2.23}$$

$$\frac{\partial v}{\partial t} + \zeta u + \dot{s}\frac{\partial v}{\partial s} + \frac{\partial}{r\partial \phi}[M + \tfrac{1}{2}(u^2 + v^2)] = F_\phi, \tag{2.24}$$

where ζ is the vertical component of the vorticity vector. Taking $-\partial(\ )/r\partial\phi$ of (2.23) and $\partial r(\ )/r\partial r$ of (2.24), then adding the results, we obtain

$$\frac{D\zeta}{Dt} + \zeta\left(\frac{\partial(ru)}{r\partial r} + \frac{\partial v}{r\partial\phi}\right) = \left(\xi\frac{\partial}{\partial r} + \eta\frac{\partial}{r\partial\phi}\right)\dot{s} + \frac{\partial(rF_\phi)}{r\partial r} - \frac{\partial F_r}{r\partial\phi}, \tag{2.25}$$

where

$$(\xi, \eta, \zeta) = \left(-\frac{\partial v}{\partial s}, \frac{\partial u}{\partial s}, f + \frac{\partial(rv)}{r\partial r} - \frac{\partial u}{r\partial\phi}\right) \tag{2.26}$$

is the absolute vorticity vector. Equation (2.26) gives the isentropic form of the vorticity equation, which indicates that the Lagrangian time rate of change of the vertical component of vorticity is related to the horizontal divergence or convergence of the flow field, the twisting of the horizontal vorticity to the vertical and the curl of the horizontal external forcing. This equation can also be written in flux form

$$\frac{\partial(\sigma P)}{\partial t} + \frac{\partial(r(u\sigma P - \xi\dot{s} - F_\phi))}{r\partial r} + \frac{\partial(v\sigma P - \eta\dot{s} + F_r)}{r\partial\phi} = 0, \tag{2.27}$$

where $P = \zeta/\sigma$ is the potential vorticity. In deriving (2.27) from (2.25), we have used the fact that the divergence of the curl of any vector field identically vanishes, i.e.,

$$\frac{\partial(r\xi)}{r\partial r} + \frac{\partial\eta}{r\partial\phi} + \frac{\partial\zeta}{\partial s} = 0. \tag{2.28}$$

Equation (2.27) is the equivalent form of the Haynes-McIntyre theorem (Haynes and McIntyre, 1987) in cylindrical coordinates. It states that the potential vorticity can not be transported across any isentropic surface because the component of flux normal to any isentrope is identically zero. In this sense, an isentropic surface is impermeable to potential vorticity. Another important concept following logically from (2.27) is that since there is no any other apparent source that can generate potential vorticity besides the horizontal PV transports: potential vorticity can neither be created nor destroyed within a layer bounded by two isentropic surfaces. Therefore, in order for the mass-weighted potential vorticity to remain unchanged, the potential vorticity must be redistributed within two isentropic surfaces as mass flows in and out. Note that the above theorem holds regardless of whether or not diabatic heating and frictional or other forces are included in the dynamic system.

The Rossby-Ertel potential vorticity principle is obtained by eliminating the horizontal divergence between (2.25) and (2.11), which yields

$$\frac{DP}{Dt} = \frac{1}{\sigma}\left[\xi\frac{\partial \dot{s}}{\partial r} + \eta\frac{\partial \dot{s}}{r\partial \phi} + \zeta\frac{\partial \dot{s}}{\partial s} + \frac{\partial(rF_\phi)}{r\partial r} - \frac{\partial F_r}{r\partial \phi}\right]. \tag{2.29}$$

Equation (2.29) indicates that the time evolution and the spatial distribution of potential vorticity can be calculated from the Lagrangian history of the diabatic heating and the frictional processes. In the absence of the diabatic heating and frictional forces potential vorticity acts as a materially conserved dynamic tracer.

In balanced dynamics, which is the main theme of this study, it turns out that the reciprocal of the Rossby-Ertel potential vorticity is a more amenable quantity to use. Let us define this reciprocal of potential vorticity $\sigma^* = f/P$ as the potential pseudodensity. It is so named because when substituting the definition of P, we have

$$\sigma^* = \frac{f}{\zeta}\sigma, \tag{2.30}$$

i.e., the potential pseudodensity is the pseudodensity (in the entropy coordinate) that an air parcel would have if its shape were changed in such a way that its vertical component of absolute vorticity took the value of the earth's vorticity. The substitution of this definition into (2.29) leads to the potential pseudodensity equation

$$\frac{D\sigma^*}{Dt} + \frac{\sigma^*}{\zeta}\left[\xi\frac{\partial \dot{s}}{\partial r} + \eta\frac{\partial \dot{s}}{r\partial \phi} + \zeta\frac{\partial \dot{s}}{\partial s} + \frac{\partial(rF_\phi)}{r\partial r} - \frac{\partial F_r}{r\partial \phi}\right] = 0, \tag{2.31}$$

which, in this case (f-plane), retains the same conservative property as that of the potential vorticity equation, i.e., the potential pseudodensity can also be treated as a materially conserved quantity when the diabatic heating and frictional forces are neglected.

2.2 The mixed geostrophic-gradient balance theory

We now begin to develop a three-dimensional balanced vortex theory. This theory generalizes Eliassen's axisymmetric balanced equations by considering the transient development of gradient balanced states. It can also be considered as a generalization of the semigeostrophic equations by inclusion of flow curvature. The following three subsections construct the first part of the theoretical framework. We first conduct a small Rossby number analysis, and through this analysis we demonstrate how the primitive system (2.8)–(2.12) can be approximated by a set of the geostrophic-gradient balanced equations. We then derive a set of conservation laws associated with these equations in order to show that the approximated system is physically valid.

2.2.1 The small Rossby number analysis

Following Hoskins (1975), we consider a frictionless motion whose horizontal projection is expressed in a natural coordinate system. The components of the momentum equation tangential (τ) and normal (n) to the direction of the motion are

$$\frac{DV}{Dt} + \frac{\partial \phi}{\partial \tau} = 0, \tag{2.32}$$

$$\frac{V^2}{r} + fV + \frac{\partial \phi}{\partial n} = 0, \tag{2.33}$$

where r is the local radius of curvature, the momentum vector is $\boldsymbol{v} = (V, 0)$, and the total acceleration vector is $D\boldsymbol{v}/Dt = (DV/Dt,\ V^2/r)$, which is composed of both inertial acceleration DV/Dt and the noninertial acceleration V^2/r. (Note: the inertial forces are defined differently in atmospheric science and in physics. In physics, the inertial force is defined as any force that results from relative motion. Under this definition, the Coriolis force and the centrifugal force are inertial forces.)

In contrast to Hoskins (1975) in which the author defined a generalized small Rossby number by

$$R_o = \frac{\left|\frac{D\boldsymbol{v}}{Dt}\right|}{|f\boldsymbol{v}|} \ll 1$$

and imposed the somewhat stringent requirement that both the inertial and noninertial accelerations are bounded by such smallness, i.e.,

$$\frac{DV}{Dt} \ll fV$$

and

$$\frac{V}{fr} \ll 1,$$

we consider a broader class of motions in which r can be small, i.e., the flow may be highly curved. This relaxation implies a reasonable redefinition of the generalized small Rossby number to be

$$R_o = \frac{\left|\frac{DV}{Dt}\right|}{|fV|} \ll 1 \tag{2.34}$$

so that

$$-\frac{V^2}{r} - \frac{\partial \phi}{\partial n} = fV \gg \frac{DV}{Dt} = -\frac{\partial \phi}{\partial \tau} \tag{2.35}$$

by taking account of (2.32) and (2.33) simultaneously. Note that the condition (2.35) differs from that of Hoskins (1975) (Eq. 8) in that the curvature effect comes into the balance relation [on the left hand side of (2.35)], and therefore the approximate equal sign in Hoskins' condition is now replaced by the exact equal sign in (2.35).

In fact, from the analysis of Hoskins (1975), two kinds of Rossby numbers were revealed in his generalized definition. The one commonly defined in meteorology and fluid mechanics is given in (2.34), which measures the relative importance of the inertial and Coriolis forces. The smallness of this Rossby number implies certain types of fluid motions with combined time and length scales characteristic of balanced adjustment processes [Note here that we are avoiding use the term "geostrophic adjustment" with the intention of not precluding more accurate adjustment processes such as gradient adjustment or even higher order adjustment. The same usage of this type of terminology has been found in McIntyre and Norton (1991).] are occuring and the gravity-inertia oscillations are not shaping the flow pattern significantly.

The second Rossby number may be called the curvature Rossby number because it can be written in the form

$$R_{oc} = \frac{V}{fr} = \frac{V^2/r}{fV},$$

which is the measure of the relative importance of the centrifugal force due to flow curvature versus the Coriolis force. In general, the smallness of this curvature Rossby number means the flow is not highly curved. It seems reasonable that in many physical situations large curvature Rossby number will not prevent the realization of balanced states.

The condition (2.34), or (2.35), is frequently met in many physical situations. For example, in a hurricane the tangential particle acceleration may be trivial while the substantial Coriolis turning is balanced by the centripetal acceleration and the radial pressure gradient force.

Since DV/Dt is small in comparison with the Coriolis force, so is $\partial\phi/\partial\tau$ then, compared to each term in (2.33), we may regard the momentum vector $(V, 0)$ is approximately balanced by the gradient momentum in the tangential direction and by the geostrophic momentum in the radial direction, i.e.,

$$(V,\, 0) = \left(-\frac{1}{f}\frac{\partial\phi}{\partial n} - \frac{V^2}{fr},\ \frac{1}{f}\frac{\partial\phi}{\partial\tau}\right). \tag{2.36}$$

This generalizes the geostrophic momentum approximation of the Eliassen type (Eliassen, 1948; Hoskins, 1975) to the geostrophic-gradient momentum approximation. The latter can treat more general flows with substantial curvature, as opposed to the former that strictly confines its applications to the quasi-straight flows, such as fronts or jets.

The above modification of Hoskins' scaling argument may be closely related to the anisotropic nature of fluid motions in the sense that the curvature effect can be completely depicted by one component of the flow field. The analysis, therefore, is reasonably two-parameter, both the Rossby number and the curvature Rossby number. Multi-scale, multi-parameter analysis procedures were also presented by McWilliams and Gent (1980), who used both the Rossby number and a frontal structure parameter, and by Allen (1991) who used both the Rossby number and a bottom topography parameter in order to arrive at the balanced equations for different applications.

While the natural coordinate system is helpful to illustrate basic ideas, the above argument may not be limited to a certain coordinate system. This approximation should be applicable to any curvilinear flow system. In fact, as we will see, we are going to apply this approximation to a curvilinear cylindrical coordinate system in the current chapter, and to a curvilinear spherical coordinate system in Chapter 3.

2.2.2 The governing equations with the geostrophic-gradient momentum approximation

Making use of the geostrophic-gradient approximation from the analysis of the small Rossby number above, we can write the set of approximated primitive equations expressed in the cylindrical and entropy coordinates (r, ϕ, s) as

$$\frac{Du_g}{Dt} - \left(f + \frac{v_g}{r}\right)\frac{v}{\gamma} + \frac{\partial M}{\gamma \partial r} = 0, \tag{2.37}$$

$$\frac{Dv_g}{Dt} + \left(f + \frac{v_g}{r}\right)u + \frac{\partial M}{r\partial \phi} = 0, \tag{2.38}$$

$$\frac{\partial M}{\partial s} = T, \tag{2.39}$$

$$\frac{D\sigma}{Dt} + \sigma\left(\frac{\partial(ru)}{r\partial r} + \frac{\partial v}{r\partial \phi} + \frac{\partial \dot{s}}{\partial s}\right) = 0, \tag{2.40}$$

where, for simplicity, the external body forces have been neglected, and most of the notations have been defined previously. Even so, we would like to reemphasize that

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + v\frac{\partial}{r\partial \phi} + \dot{s}\frac{\partial}{\partial s} \tag{2.41}$$

is the total derivative, and (u, v) the total radial and tangential components of the velocity, while u_g the geostrophic radial wind defined by

$$-fu_g = \frac{\partial M}{R\partial \phi}, \tag{2.42}$$

and v_g the gradient tangential wind, i.e.,

$$\left(f + \frac{v_g}{r}\right)v_g = \frac{\partial M}{\partial r}. \tag{2.43}$$

Note that R is different from r and is the potential radius, which will be defined later. The parameter γ is naturally brought into existence mathematically due to the geostrophic modification of total momentum in a curvilinear system in (2.37). It can be regarded as a parameter which measures the relative importance of the curvature vorticity with respect to f. This can be understood from its definition:

$$\gamma = \left(\frac{r}{R}\right)\frac{f + v_g/r}{f}. \tag{2.44}$$

When the curvature vorticity is small compared with f, R reduces to r [ref. (2.55)] and $\gamma = 1$. Then the whole system reduces to the semigeostrophic set of equations. If we further replace (u_g, v_g) by their full counterparts (u, v), (2.34)–(2.37) revert to the primitive equations. It will be shown later that the canonical transformations of the new balanced equations (2.37)–(2.43) are crucially dependent upon appearance of this γ factor in the dynamical system.

Careful examination of γ as the function of the curvature vorticity, by substituting the formula for R presented in (2.55) of the next section, gives

$$\gamma = \frac{1 + R_{oc}}{\sqrt{2R_{oc} + 1}}, \tag{2.45}$$

where R_{oc} is the curvature Rossby number, which has been previously defined as

$$R_{oc} = \frac{v_g/r}{f}.$$

Figure 2.1 shows γ as a function of curvature vorticity normalized by the earth's vorticity f (i.e., the curvature Rossby number). When curvature vorticity is absent in the flow field ($R_{oc} = 0$), $\gamma = 1$. Note that there is considerable departure of γ from its quasi-straight flow value ($\gamma = 1$) described by semigeostrophic equations as the absolute value of the curvature vorticity increases. There is a run-away effect for γ as the vortex spins up. For example, the value of γ is increased 16% when $v_g/r = f$, 34% when $v_g/r = 2f$ and 140% when $v_g/r = 10f$. In fully a developed hurricane, the curvature vorticity frequently exceeds $10f$ (Hawkins and Imbembo, 1976; Sheets, 1980). It is also found that there is a cut-off value of $v_g/r = -\frac{1}{2}f$ for anticyclonic flows, in which case $\gamma \to \infty$ and this theory breaks down, which may be related to inertial instability of the flow.

Since the total momentum in the Lagrangian acceleration and curvature terms is approximated either by the geostrophic or by the gradient wind, we shall henceforth refer to (2.37)–(2.43) as the mixed-balance system. By the "balanced system" we mean that due to the balanced assumptions (2.42) and (2.43), the space constituted by the linear eigenmodes of this system has been shrunk to the subspace comprised only by the slow manifold. In other words, the approximated system no longer permits transient

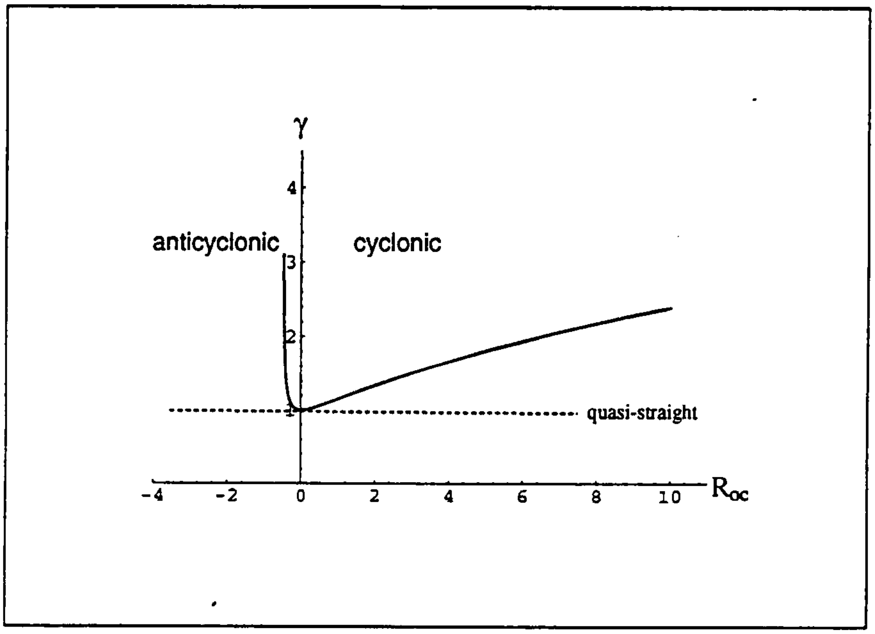


Figure 2.1: γ plotted as the function of the curvature vorticity (curvature Rossby number). Semigeostrophic and quasi-geostrophic theories are only valid at γ=1.

gravitational modes as part of its possible solutions. Together with the equation of the state, (2.37)–(2.43) form a closed system, i.e., there are seven equations for seven unknowns u, v, u_g, v_g, M, T and σ (the formula that defines the potential radius will be given in the next section). Note that u_g and v_g are diagnosed from M through (2.42) and (2.43), and M is related to σ (although the relation is not so obvious at this point, we will prove later in section 2.5 that such an invertibility principle does exist in transformed space), while σ is already predicted by (2.40). Therefore, (2.37) and (2.38) are not independent predictors. In this regard, only one prognostic equation is left, i.e.,, the system has only one class of eigenfrequencies.

2.2.3 The conservation principles

The above mixed geostrophic and gradient balanced system preserves the following physical laws:

a. Angular momentum conservation

We define the absolute gradient angular momentum as

$$m_g = rv_g + \tfrac{1}{2}fr^2. \tag{2.46}$$

On taking the material derivative of (2.46) and using (2.38), we indeed have the angular momentum principle

$$\frac{Dm_g}{Dt} + \frac{\partial M}{\partial \phi} = 0. \tag{2.47}$$

When an axisymmetric flow is considered, the angular momentum m_g becomes a materially conserved quantity. Such a conservation principle implies that there must exist a cyclic coordinate that can be used to transform the momentum equation into a canonical form. In fact, as we will discuss later, $m_g = \frac{1}{2}fR^2$ defines the potential radius.

Using the continuity equation, (2.47) can be written in the flux form

$$\frac{\partial(\sigma m_g)}{\partial t} + \frac{\partial(\sigma r u m_g)}{r\partial r} + \frac{\partial(\sigma v m_g)}{r\partial \phi} + \frac{\partial(\sigma \dot{s} m_g)}{\partial s} + \sigma\frac{\partial M}{\partial \phi} = 0. \tag{2.48}$$

By applying the massless layer approach, we may integrate this equation both vertically and azimuthaly to obtain

$$\frac{\partial}{\partial t}\iint m_g \sigma r d\phi ds + \frac{\partial}{\partial r}\iint m_g u \sigma r d\phi ds + \iint \frac{\partial M}{\partial \phi}\sigma r d\phi ds = 0, \tag{2.49}$$

which is the approximate form of the integrated angular momentum conservation principle (2.16) in the sense that the total angular momentum is now replaced by the gradient angular momentum.

b. Energy conservation

Adding u_g times (2.37) and v_g times (2.38), the kinetic energy equation becomes

$$\frac{DK_g}{Dt} + u\frac{\partial M}{\partial r} + v\frac{\partial M}{r\partial \phi} = 0, \tag{2.50}$$

where $K_g = \frac{1}{2}(u_g^2 + v_g^2)$ is the combined geostrophic-gradient kinetic energy. Following the same procedures as for the primitive system of equations, we can write this equation in the flux form

$$\begin{aligned} &\frac{\partial}{\partial t}(\sigma K_g) + \frac{\partial}{r\partial r}(r\sigma u(K_g + gz)) + \frac{\partial}{r\partial \phi}(\sigma v(K_g + gz)) \\ &\qquad + \frac{\partial}{\partial s}\left(\sigma\dot{s}(K_g + gz) - gz\frac{\partial p}{\partial t}\right) + \sigma\alpha\omega = 0. \end{aligned} \tag{2.51}$$

The thermodynamic energy equation can be written, after using the mass continuity equation (2.40), in the form

$$\frac{\partial}{\partial t}(\sigma c_p T) + \frac{\partial}{r\partial r}(r\sigma u c_p T) + \frac{\partial}{r\partial \phi}(\sigma v c_p T) + \frac{\partial}{\partial s}(\sigma\dot{s}c_p T) - \sigma\alpha\omega = \sigma Q, \tag{2.52}$$

where α, ω and Q are all defined in the same way as they were in (2.19) and (2.20). Again, the addition of (2.51) and (2.52) results in the cancellation of the conversion term $\sigma\alpha\omega$ and leads to a total energy equation

$$\begin{aligned} &\frac{\partial}{\partial t}(\sigma(K_g + c_p T)) + \frac{\partial}{r\partial r}(r\sigma u(K_g + M)) + \frac{\partial}{r\partial \phi}(\sigma v(K_g + M)) \\ &\qquad + \frac{\partial}{\partial s}\left(\sigma\dot{s}(K_g + M) - gz\frac{\partial p}{\partial t}\right) = \sigma Q. \end{aligned} \tag{2.53}$$

Assuming the top boundary is both an isentropic and isobaric surface, assuming no topography and vanising $\dot{s}$ at the top and bottom, we integrate the total energy equation to obtain

$$\frac{\partial}{\partial t}\iint (K_g + c_p T)\sigma r d\phi ds + \frac{\partial}{\partial r}\iint (K_g + M)u\sigma r d\phi ds = \iint Q\sigma r d\phi ds, \tag{2.54}$$

where we have taken into account cases when the lower isentropes terminate at the earth's surface by adopting the massless layer approach. In comparing (2.54) with (2.22), we see that, except for the fact that the kinetic energy is evaluated using u_g and v_g, the approximated governing equations (2.37)–(2.41) have a total energy conservation principle identical to the one which exists for the primitive equations.

We will delay the derivations of the vorticity and the potential vorticity equations until section 2.4 after a discussion of the coordinate transformation.

2.3 The combined geostrophic azimuth, potential radius and entropy coordinate transformations

As discussed in the previous section, the implementation of mixed geostrophic and gradient momentum approximations in the primitive equations leads to a set of balanced equations in which three prognostic equations have been reduced to one. Although this balanced system is closed, it is not a mathematically convenient set to work with, and some kind of reformulation procedure is necessary. This is exactly the same situation as in semigeostrophic theory. Semigeostrophic theory has been pushed into a viable position due to two devices (Hoskins, 1975): (1) the geostrophic approximations; (2) the geostrophic coordinate transformations. Through the latter, the geostrophically approximated equations of the Eliassen type may systematically be transformed to their canonical forms, and the further use of these canonical equations makes the ageostrophic part of the total wind implicit in the coordinate transformation. Hence, the system of equations reduces to an amenable mathematical formulation: one predictive equation for potential vorticity and one diagnostic invertibility principle to obtain the balanced wind and mass fields. This methodology is also valid in the axisymmetric vortex model [Shutts and Thorpe (1978) and Schubert and Hack (1983)] where another type of quasi-Lagrangian coordinate, the so-called potential radius, is used. Another issue concerns the duality between the use of the geostrophic coordinates in the horizontal and the isentropic coordinate in the vertical. This duality was first pointed out by Hoskins and Draghici (1977), and has been further discussed by Gill (1981) and Heckley and Hoskins (1982). The possibility of the combined use of geostrophic and isentropic coordinates has been discussed theoretically by McWilliams and Gent (1980). Buzzi *et al.* (1981) studied the two dimensional internal frontogenesis problem by simultaneously using the geostrophic coordinates and the isentropic coordinate. This approach has been extended to the three dimensional semi-geostrophic system on the f-plane (Schubert *et al.*, 1989), β-plane (Magnusdottir and Schubert, 1990) and hemisphere (Magnusdottir and Schubert, 1991). Here it is logical for us to combine the geostrophic coordinate, potential radius and entropy as one set of quasi-Lagrangian coordinates to transform our mixed geostrophic and gradient balanced system.

The fact that the mixed balance system preserves the absolute angular momentum provides us the choice of a new coordinate, the potential radius, as the replacement of the physical radius. It is defined by

$$\tfrac{1}{2} f R^2 = r v_g + \tfrac{1}{2} f r^2, \tag{2.55}$$

i.e., the potential radius R is the radius to which an air particle must be moved (conserving its absolute angular momentum) in order for its relative angular momentum to vanish (Schubert and Hack, 1983).

Since we assumed that the radial momentum is in geostrophic balance, it is natural to introduce the geostrophic azimuth, that is, the azimuth air particles would have if they were moved with their geostrophic angular velocity at every instant. Mathematically, this relation can be written

$$\Phi = \phi - \frac{u_g}{fR}. \tag{2.56}$$

Combining the potential radius and geostrophic azimuth as two new coordinates, (2.37)–(2.38) can be so transformed that the horizontal ageostrophic winds become completely implicit. In addition, let us define $S = s$ and $\mathcal{T} = t$, but noting that $\partial/\partial s$ and $\partial/\partial t$ imply fixed r, ϕ while $\partial/\partial S$ and $\partial/\partial \mathcal{T}$ imply fixed R, Φ. With these newly defined coordinates, we can now proceed to transform our balanced system (2.37)–(2.40) from (r, ϕ, s, t) space to $(R, \Phi, S, \mathcal{T})$ space. The derivative relations in the two spaces are given by

$$\frac{\partial}{\partial t} = \frac{\partial R}{\partial t}\frac{\partial}{\partial R} + \frac{\partial \Phi}{\partial t}\frac{\partial}{\partial \Phi} + \frac{\partial}{\partial \mathcal{T}}, \tag{2.57}$$

$$\frac{\partial}{\partial r} = \frac{\partial R}{\partial r}\frac{\partial}{\partial R} + \frac{\partial \Phi}{\partial r}\frac{\partial}{\partial \Phi}, \tag{2.58}$$

$$\frac{\partial}{\partial\phi}=\frac{\partial R}{\partial\phi}\frac{\partial}{\partial R}+\frac{\partial\Phi}{\partial\phi}\frac{\partial}{\partial\Phi},\tag{2.59}$$

$$\frac{\partial}{\partial s}=\frac{\partial R}{\partial s}\frac{\partial}{\partial R}+\frac{\partial\Phi}{\partial s}\frac{\partial}{\partial\Phi}+\frac{\partial}{\partial S}.\tag{2.60}$$

Applying this set to the Bernoulli function $M^*=M+\frac{1}{2}(u_g^2+v_g^2)$, we can prove that

$$\left(\frac{\partial M}{\gamma\partial r},\ \frac{\partial M}{\partial\phi},\ \frac{\partial M}{\partial s},\ \frac{\partial M}{\partial t}\right)=\left(\frac{\partial M^*}{\partial R}-\frac{u_g^2}{R},\ \frac{\partial M^*}{\partial\Phi},\ \frac{\partial M^*}{\partial S},\ \frac{\partial M^*}{\partial T}\right).\tag{2.61}$$

The transformation relations (2.57)–(2.60) also imply that the total derivative (2.41) can be written as

$$\frac{D}{Dt}=\frac{\partial}{\partial T}+U\frac{\partial}{\partial R}+V\frac{\partial}{R\partial\Phi}+\dot{S}\frac{\partial}{\partial S},\tag{2.62}$$

where

$$\left(U,\ V,\ \dot{S}\right)=\left(\frac{DR}{Dt},\ R\frac{D\Phi}{Dt},\ \frac{DS}{Dt}\right),\tag{2.63}$$

is the vector velocity in transformed space, with $\dot{s}=\dot{S}$.

With the aid of (2.61) we can now show that (2.37) and (2.38) take the canonical forms (the detailed proof is given in Appendix A)

$$fR\frac{D\Phi}{Dt}=\frac{\partial M^*}{\partial R},\tag{2.64}$$

$$-f\frac{DR}{Dt}=\frac{\partial M^*}{R\partial\Phi}.\tag{2.65}$$

It is interesting to note that the horizontal advective winds in the Lagrangian time derivative (2.62) are related to the Bernoulli function in such a way that they are formally in geostrophic balance. These advecting velocities are solely determined by geostrophic and gradient winds in physical space through (2.61), (2.42) and (2.43). Therefore, the major advantages of the coordinate transformation from (r,ϕ,s,t) space to (R,Φ,S,T) space are that the two momentum equations are reduced to their canonical forms and the substitutions of these canonical equations into (2.62) result in the absence of ageostrophic advection. In addition, for adiabatic motions the vertical advection does not appear in (2.62), so that the predicted motions becoming quasi-horizontal in such a coordinate space.

2.4 Vorticity, potential vorticity and potential pseudodensity equations

In the nondivergent, barotropic atmosphere, the flow can be described entirely in terms of the vorticity field (Rossby, 1939; Hoskins *et al.*, 1985). As a matter of fact, in such an idealized model the dynamics and the thermodynamics are decoupled so that the vorticity field recovers all the relevant information regarding the fluid motion. In a more realistic stratified, divergent, baroclinic atmosphere, however, both the dynamic and thermodynamic aspects are indispensable in determining fluid motion. In this case, a more general dynamic tracer and predictive equation are needed. Such a succinct dynamic statement is given by the Rossby-Ertel potential vorticity principle (Rossby, 1940; Ertel, 1942). In the primitive system, the significance of this principle lies merely in that it provides an additional conservation law for the third Lagrangian marker necessary to identify an air parcel, and since the mass and wind fields are explicitly predictive quantities in the primitive equations, potential vorticity is not an essential concept understanding dynamic processes. For the balanced system, however, there is another important principle accompaning the potential vorticity principle, i.e., the invertibility principle (Hoskins *et al.*, 1985). The two principles work co-operatively in such a way that the potential vorticity principle can be used as the fundamental prognostic equation and the invertibility principle as the fundamental diagnostic equation. While the invertibility principle is the subject of the next section, we discuss the potential vorticity principle associated with our mixed-balance system in the current section. The two-fold purpose in presenting the potential vorticity principle here is that it serves as the fundamental prognostic equation, and at the same time, it concludes the conservation argument begun in section 2.2.

We first discuss the vorticity equation associated with our mixed-balance system. The simplest way to derive the vorticity equation is to combine the derivative of (2.64) and (2.65) in such a way as to form the total derivative of $f\partial(\frac{1}{2}R^2, \Phi)/\partial(\frac{1}{2}r^2, \phi)$, i.e., to form $(\frac{1}{2}R^2)_{\frac{1}{2}r^2}[(2.64)/R]_\phi - \Phi_\phi[(2.65)R]_{\frac{1}{2}r^2} - (\frac{1}{2}R^2)_\phi[(2.64)/R]_{\frac{1}{2}r^2} + \Phi_{\frac{1}{2}r^2}[(2.65)R]_\phi$. In doing so, we obtain (see details in Appendix B)

$$\frac{D\zeta_g}{Dt} + \zeta_g\left(\frac{\partial(ru)}{r\partial r} + \frac{\partial v}{r\partial\phi}\right) - \left(\xi_g\frac{\partial}{\partial r} + \eta_g\frac{\partial}{r\partial\phi}\right)\dot{s} = 0, \tag{2.66}$$

where

$$(\xi_g, \eta_g, \zeta_g) = f\left(\frac{\partial(\frac{1}{2}R^2, \Phi)}{r\partial(\phi, s)}, \frac{\partial(\frac{1}{2}R^2, \Phi)}{\partial(s, r)}, \frac{\partial(\frac{1}{2}R^2, \Phi)}{\partial(\frac{1}{2}r^2, \phi)}\right) \tag{2.67}$$

is the vector vorticity associated with the geostrophic and gradient winds. If we define $P_g = \zeta_g/\sigma$ as the balanced version of Rossby-Ertel potential vorticity, (2.66) can be written in the flux form

$$\frac{\partial(\sigma P_g)}{\partial t} + \frac{\partial(r(u\sigma P_g - \xi_g \dot{s}))}{r\partial r} + \frac{\partial(v\sigma P_g - \eta_g \dot{s})}{r\partial\phi} = 0, \tag{2.68}$$

by virtue of the vector identity

$$\frac{\partial(r\xi_g)}{r\partial r} + \frac{\partial\eta_g}{r\partial\phi} + \frac{\partial\zeta_g}{\partial s} = 0. \tag{2.69}$$

Equation (2.68) is the balanced version of (2.27). Thus, the Haynes-McIntyre theorem is naturally preserved in our mixed balanced system.

Before we derive the potential vorticity principle, we would like first to show the following useful relations. From (2.58)-(2.59) we may have

$$\frac{\partial(\frac{1}{2}R^2, \Phi)}{\partial(\frac{1}{2}r^2, \phi)} \frac{\partial}{R\partial R} = \frac{\partial\Phi}{\partial\phi} \frac{\partial}{r\partial r} - \frac{\partial\Phi}{r\partial r} \frac{\partial}{\partial\phi}, \tag{2.70}$$

$$\frac{\partial(\frac{1}{2}R^2, \Phi)}{\partial(\frac{1}{2}r^2, \phi)} \frac{\partial}{\partial\Phi} = -\frac{R\partial R}{\partial\phi} \frac{\partial}{r\partial r} + \frac{R\partial R}{r\partial r} \frac{\partial}{\partial\phi}. \tag{2.71}$$

In making use of (2.70) and (2.71) in (2.60) we can prove that

$$\xi_g \frac{\partial}{\partial r} + \eta_g \frac{\partial}{r\partial\phi} + \zeta_g \frac{\partial}{\partial s} = \zeta_g \frac{\partial}{\partial S}. \tag{2.72}$$

This relation shows that $\partial/\partial S$ is actually the derivative along the vorticity vector, and that is why we sometimes refer to (R, Φ, S, T) as "vortex coordinates".

The potential vorticity equation is derived by combining the vorticity equation (2.66) and the continuity equation (2.40). The result is

$$\sigma \frac{DP_g}{Dt} = \left(\xi_g \frac{\partial}{\partial r} + \eta_g \frac{\partial}{r\partial\phi} + \zeta_g \frac{\partial}{\partial s}\right) \dot{s} = \zeta_g \frac{\partial \dot{s}}{\partial S}, \tag{2.73}$$

where $P_g = \zeta_g/\sigma$ is the combined geostrophic-gradient balanced potential vorticity. Equation (2.73) states that the potential vorticity is a conserved quantity following fluid particles when diabatic heating is absent ($\dot{s} = 0$). This statement, of course, is the generalization of Ertel's theorem for the 3-D balanced vortex model.

We next derive the equation for the inverse of the potential vorticity, i.e., the potential pseudodensity equation. The advantage of using potential pseudodensity as the primary prognostic quantity has been pointed out before, and will be seen more clearly in the next section. In an analogous way to the discussion of the primitive equations, let us define the potential pseudodensity as

$$\sigma_g^* = \frac{f}{\zeta_g}\sigma \tag{2.74}$$

so that the potential vorticity P_g and the potential pseudodensity σ_g^* are related by $P_g\sigma_g^* = f$. On substituting this definition into (2.73), we obtain

$$\frac{D\sigma_g^*}{Dt} + \sigma_g^*\frac{\partial \dot{s}}{\partial S} = 0. \tag{2.75}$$

When the diabatic heating is absent, potential pseudodensity is also materially conserved. By using (2.62)–(2.63), we can write the potential pseudodensity equation in the flux form

$$\frac{\partial\sigma_g^*}{\partial T} + \frac{\partial(RU\sigma_g^*)}{R\partial R} + \frac{\partial(V\sigma_g^*)}{R\partial\Phi} + \frac{\partial(\dot{S}\sigma_g^*)}{\partial S} = 0, \tag{2.76}$$

where U and V in the horizontal flux terms are given in (2.63) and are related to the single variable M^* through (2.64) and (2.65). $\dot{S} = \dot{s}$ is the diabatic heating, which is either specified or given by some kind of parameterization. Thus, the integration of (2.76) forward in time requires only the initial σ_g^* field and the history of the diabatic heating provided that M^* is somehow obtainable, which is the topic of the next section.

2.5 Invertibility principle

In order to complete the predictive cycle, we shall next search for a diagnostic equation which can invert the predicted σ_g^* to yield the basic diagnostic variable M^*. We begin with the definition of σ_g^* (2.74), which can be written as

$$\frac{\partial(\frac{1}{2}r^2,\phi)}{\partial(\frac{1}{2}R^2,\Phi)}\frac{\partial p}{\partial s} + \sigma_g^* = 0, \tag{2.77}$$

by noting (2.67). Applying (2.60) to $\frac{1}{2}r^2$ and ϕ respectively, and combining the two resultant equations to yield the two relations

$$\frac{R\partial R}{\partial s}\frac{r\partial r}{R\partial R}\frac{\partial\phi}{\partial\Phi} - \frac{R\partial R}{\partial s}\frac{\partial\phi}{R\partial R}\frac{r\partial r}{\partial\Phi} = \frac{\partial\phi}{\partial S}\frac{r\partial r}{\partial\Phi} - \frac{r\partial r}{\partial S}\frac{\partial\phi}{\partial\Phi}, \tag{2.78}$$

$$\frac{\partial\Phi}{\partial s}\frac{r\partial r}{\partial\Phi}\frac{\partial\phi}{R\partial R}-\frac{\partial\Phi}{\partial s}\frac{\partial\phi}{\partial\Phi}\frac{r\partial r}{R\partial R}=\frac{\partial\phi}{\partial S}\frac{r\partial r}{R\partial R}-\frac{r\partial r}{\partial S}\frac{\partial\phi}{R\partial R}, \tag{2.79}$$

we then substitute (2.78) and (2.79) in the expansion of the first term in (2.77) to obtain

$$\begin{aligned}\frac{\partial(\frac{1}{2}r^2,\phi)}{\partial(\frac{1}{2}R^2,\Phi)}\frac{\partial p}{\partial s} &= \frac{\partial\phi}{\partial S}\frac{r\partial r}{\partial\Phi}\frac{\partial p}{R\partial R}-\frac{r\partial r}{\partial S}\frac{\partial\phi}{\partial\Phi}\frac{\partial p}{R\partial R}+\frac{\partial\phi}{\partial S}\frac{r\partial r}{R\partial R}\frac{\partial p}{\partial\Phi}\\ &-\frac{r\partial r}{\partial S}\frac{\partial\phi}{R\partial R}\frac{\partial p}{\partial\Phi}+\frac{r\partial r}{R\partial R}\frac{\partial\phi}{\partial\Phi}\frac{\partial p}{\partial S}-\frac{r\partial r}{\partial\Phi}\frac{\partial\phi}{R\partial R}\frac{\partial p}{\partial S}\\ &=\frac{\partial(\frac{1}{2}r^2,\phi,p)}{\partial(\frac{1}{2}R^2,\Phi,S)},\end{aligned}$$

which leads to the invertibility principle in the Jacobian form

$$\frac{\partial(\frac{1}{2}r^2,\phi,p)}{\partial(\frac{1}{2}R^2,\Phi,S)}+\sigma_g^*=0, \tag{2.80}$$

where now r, ϕ, p and σ_g^* are all expressed as dependent variables in (R,Φ,S,T) space. This is the essential advantage of using the potential pseudodensity as the fundamental predictive quantity.

We notice that the additional term in the first entry of (2.61) presents a small correction to the mixed geostrophic and gradient balanced flow in an asymmetric vortex. This can be easily seen by comparing this term to the term on the left hand side of (2.61), i.e.,

$$\frac{\left|u_g^2/R\right|}{|\partial M/\gamma\partial r|}=\frac{u_g^2}{|2v_g^2+frv_g|}<\frac{u_g^2}{2v_g^2} \tag{2.81}$$

for a cyclonic vortex. This quantity should be very small compared to unity for most vortex circulations such as hurricanes, where the tangential wind can be one order of magnitude larger than the radial wind so that the numerator is approximately 200 times smaller than the denominator. For this reason this small additional term can be dropped, which results in the geostrophic, gradient and hydrostatic balanced relations in the transformed space taking the forms

$$(fv_g,-fu_g,T)=\left(\frac{r}{R}\frac{\partial M^*}{\partial R},\frac{\partial M^*}{R\partial\Phi},\frac{\partial M^*}{\partial S}\right). \tag{2.82}$$

Using this set of relations in the geostrophic azimuth coordinate (2.56), potential radius (2.55) and the ideal gas law, we can write r, ϕ and p all in terms of M^* as

$$\tfrac{1}{2}r^2 = \frac{\tfrac{1}{2}R^2}{1+\frac{2}{f^2R}\frac{\partial M^*}{\partial R}} = \Gamma(M^*), \tag{2.83}$$

$$\phi = \Phi - \frac{1}{f^2}\frac{\partial M^*}{R^2\partial \Phi}, \tag{2.84}$$

$$p = \rho \mathrm{R}\frac{\partial M^*}{\partial S}, \tag{2.85}$$

where R is the gas constant. On substituting (2.83)–(2.85) into (2.80), the invertibility principle can be rewritten as

$$\frac{1}{f^2}\begin{vmatrix} \frac{\partial \Gamma(M^*)}{R\partial R} & \frac{\partial \Gamma(M^*)}{\partial \Phi} & \frac{\partial \Gamma(M^*)}{\partial S} \\ \frac{\partial}{R\partial R}\left(\frac{\partial M^*}{R^2\partial \Phi}\right) & \frac{1}{R^2}\frac{\partial^2 M^*}{\partial \Phi^2} - f^2 & \frac{\partial}{\partial S}\left(\frac{\partial M^*}{R^2\partial \Phi}\right) \\ \frac{\partial}{R\partial R}\left(\rho \mathrm{R}\frac{\partial M^*}{\partial S}\right) & \frac{\partial}{\partial \Phi}\left(\rho \mathrm{R}\frac{\partial M^*}{\partial S}\right) & \frac{\partial}{\partial S}\left(\rho \mathrm{R}\frac{\partial M^*}{\partial S}\right) \end{vmatrix} - \sigma_g^* = 0. \tag{2.86a}$$

This is the basic diagnostic equation we are seeking; it determines M^* from the known σ_g^* field at each time step. This equation, known as a Monge-Ampère equation of elliptic type, has been extensively studied in the mathematics community both for its generalized solution and the existence of the unique solution [e.g., Pogorelov (1964), Bakelman (1957, 1958a, 1958b), Courant and Hilbert (1966)]. Equation (2.86a) presents a three dimensional form of the Monge-Ampère equation instead of the two dimensional one in the studies mentioned above. In this sense, we are dealing with a more complicated problem here. However, we conjecture that there is no substantive change of the problem in going from two dimensions to three dimensions.

Note that when we expand the determinant in this equation, it will present a three dimensional, second order, nonlinear elliptic type of partial differential equation. To integrate this equation six boundary conditions are required. The vertical boundary conditions can be derived in a similar fashion as that in Schubert *et al.* (1989). By neglecting the effects of topography and assuming that the lower boundary is the constant height surface

$z = 0$ and the isentropic surface, we conclude that $M = c_pT$ at $S = S_B$. Written in terms of M^* , this lower boundary condition becomes

$$c_p\frac{\partial M^*}{\partial S} - M^* + \frac{1}{2f^2}\left[\left(\frac{\partial M^*}{R\partial\Phi}\right)^2 + \left(\frac{\partial M^*}{\partial R}\right)^2\right] = 0 \qquad \text{at } S = S_B. \tag{2.86b}$$

The upper boundary is assumed to be an isentropic and isobaric surface, and hence also an isothermal one. Directly applying the hydrostatic equation to this surface, we can write the upper boundary condition for (2.86a) as

$$\frac{\partial M^*}{\partial S} = T_T \qquad \text{at } S = S_T, \tag{2.86c}$$

where T_T is the constant temperature at the top layer of the model domain. In practice, this upper boundary should be chosen high enough so that it will not affect the interesting physical processes inside the model domain. It is interesting to note that the vertical boundary conditions in this model are almost identical to those in the semigeostrophic theory (Schubert *et al.*, 1989; Magnusdottir and Schubert, 1991), despite the different balance assumptions employed.

Since we are developing a vortex model, we may use the periodic boundary conditions both in M^* and the derivative of M^* in azimuthal direction.

We next consider the inner radial boundary at $R = 0$. According to (2.82), in order for the tangential wind not to become singular at this point, either r or $\partial M^*/\partial R$ must be zero. But the asymmetry of the vortex requires $r \neq 0$ at $R = 0$, then it must be

$$\frac{\partial M^*}{\partial R} = 0 \qquad \text{at } R = 0. \tag{2.86d}$$

As for the outer radial boundary condition, we wish to impose a condition at finite $R = R_L$ which simulates the far field motion, in the sense that the influence of localized forcing nearly disappears as R_L becomes large. This allows one to impose a Dirichlet boundary condition

$$M^* = \psi(S) \qquad \text{at } R = R_L, \tag{2.86e}$$

by integrating the hydrostatic equation so that $\psi(S)$ is given by

$$\psi(S) = \int_{S_B}^{S}\left(\frac{p}{p_0}\right)^{\kappa}\Theta_0 exp\left(\frac{S}{c_p}\right)dS + c_pT_B, \tag{2.87}$$

where T_B is the temperature field at the bottom layer of the model domain, p the undisturbed pressure profile and $\kappa = \mathsf{R}/c_p$. The quantities with "0" subscript denote surface reference values. Given the basic state pressure as the function of the vertical coordinate, the above integral can be obtained either analytically or numerically. In this sense the Dirichlet boundary condition at $R = R_L$ is specified.

• *Summary of the mixed geostrophic-gradient balanced vortex model*

We have now completed our derivation of a mathematical model which filters transient gravity waves, and is able to describe three dimensional circular flows. The model consists of two fundamental equations: one prognostic equation (2.76) and one diagnostic equation (2.80) or (2.86). The predictive variable is the potential pseudodensity from which both the mass and wind fields are retrieved by using the invertibility principle. The summary of this balanced model is listed in Table 2.1.

2.6 Axisymmetric dynamics

In this section, we demonstrate that the Eliassen balanced vortex model (Eliassen, 1952) and the transformed Eliassen balanced vortex theory for an idealized axisymmetric flow (Shutts and Thorpe, 1978; Schubert and Hack, 1983; Schubert and Alworth, 1987) are, in fact, the two dimensional special cases of our mixed geostrophic gradient balanced theory discussed in this chapter. This result suggests that the theory presented so far is consistent with Eliassen's model.

We begin first by imposing the axisymmetry assumption in our generalized three dimensional model, that is, letting all terms involving $\partial/\partial\phi$ (or $\partial/\partial\Phi$ in the transformed space) be zero, except for the term $\partial\Phi/\partial\phi$ being unity. In doing so, from (2.42), (2.43) and (2.37), we deduce that

$$u_g = 0, \quad u = u_a, \tag{2.88}$$

$$v_a = 0, \quad v = v_g, \tag{2.89}$$

which indicate that the radial wind becomes the pure ageostrophic wind, while the azimuthal wind is purely in gradient balance. This is precisely the view of the dynamic

Table 2.1: Summary of the mixed geostrophic-gradient balanced vortex model.

$$\frac{\partial \sigma_g^*}{\partial T} + \frac{\partial (RU\sigma_g^*)}{R\partial R} + \frac{\partial (V\sigma_g^*)}{R\partial \Phi} + \frac{\partial (\dot{S}\sigma_g^*)}{\partial S} = 0, \tag{T2.1}$$

$$\frac{1}{f^2}\begin{vmatrix} \frac{\partial \Gamma(M^*)}{R\partial R} & \frac{\partial \Gamma(M^*)}{\partial \Phi} & \frac{\partial \Gamma(M^*)}{\partial S} \\ \frac{\partial}{R\partial R}\left(\frac{\partial M^*}{R^2\partial \Phi}\right) & \frac{1}{R^2}\frac{\partial^2 M^*}{\partial \Phi^2} - f^2 & \frac{\partial}{\partial S}\left(\frac{\partial M^*}{R^2\partial \Phi}\right) \\ \frac{\partial}{R\partial R}\left(\rho \mathrm{R}\frac{\partial M^*}{\partial S}\right) & \frac{\partial}{\partial \Phi}\left(\rho \mathrm{R}\frac{\partial M^*}{\partial S}\right) & \frac{\partial}{\partial S}\left(\rho \mathrm{R}\frac{\partial M^*}{\partial S}\right) \end{vmatrix} - \sigma_g^* = 0, \tag{T2.2}$$

$$c_p\frac{\partial M^*}{\partial S} - M^* + \frac{1}{2f^2}\left[\left(\frac{\partial M^*}{R\partial \Phi}\right)^2 + \left(\frac{\partial M^*}{\partial R}\right)^2\right] = 0 \quad \text{at } S = S_B, \tag{T2.3}$$

$$\frac{\partial M^*}{\partial S} = T_T \quad \text{at } S = S_T, \tag{T2.4}$$

$$\frac{\partial M^*}{\partial R} = 0 \quad \text{at } R = 0, \tag{T2.5}$$

$$M^* = \psi(S) \quad \text{at } R = R_L, \tag{T2.6}$$

configuration by the axisymmetric balanced vortex theory. With use of (2.88) and (2.89) the set of governing equations (2.37)–(2.41) is reduced to

$$\left(f+\frac{v}{r}\right)v=\frac{\partial M}{\partial r}, \tag{2.90}$$

$$\frac{Dv}{Dt}+\left(f+\frac{v}{r}\right)u=0, \tag{2.91}$$

$$\frac{\partial M}{\partial s}=T, \tag{2.92}$$

$$\frac{D\sigma}{Dt}+\sigma\left(\frac{\partial(ru)}{r\partial r}+\frac{\partial\dot{s}}{\partial s}\right)=0, \tag{2.93}$$

where the total derivative now becomes

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}+\dot{s}\frac{\partial}{\partial s}. \tag{2.94}$$

These are the s-coordinate versions of Eliassen's balanced vortex equations.

In the transformed space, again, let us impose that $\partial/\partial\Phi=0$ in the system of (T2.1)–(T2.6). The fundamental predictive equation, i.e., the potential pseudodensity equation (T2.1) is reduced to the simple form

$$\frac{\partial\sigma^*}{\partial T}+\frac{\partial(\dot{s}\sigma^*)}{\partial S}=0, \tag{2.95}$$

which is identical to the axisymmetric potential pseudodensity equation obtained by Schubert and Alworth (1987, Eq. 26) when the external forces are neglected. The simplicity of this equation lies not only in its form, but also in the computational procedure because prediction of the future σ^* field is completely decoupled from the diagnostic operation if the diabatic heating is explicitly given. Furthermore, this equation can be integrated analytically by adopting the method of characteristics, as shown in Schubert and Alworth (1987).

The reduction of the 3×3 determinant to a 2×2 one is immediately seen when the axisymmetric condition is applied in (T2.2):

$$\begin{vmatrix} \frac{\partial\Gamma(M^*)}{R\partial R} & \frac{\partial\Gamma(M^*)}{\partial S} \\ \frac{\partial}{R\partial R}\left(\rho R\frac{\partial M^*}{\partial S}\right) & \frac{\partial}{\partial S}\left(\rho R\frac{\partial M^*}{\partial S}\right) \end{vmatrix}+\sigma_g^*=0. \tag{2.96}$$

Substitution of the function $\Gamma(M^*)$ in (2.96) and expansion of the determinant gives

$$\left[f^2 - R^3\frac{\partial}{\partial R}\left(R^{-3}\frac{\partial M^*}{\partial R}\right)\right]\frac{\partial}{\partial S}\left(\rho\frac{\partial M^*}{\partial S}\right) + \frac{\partial^2 M^*}{\partial S\partial R}\frac{\partial}{\partial R}\left(\rho\frac{\partial M^*}{\partial S}\right) + \frac{\sigma_g^*}{\mathrm{R}}\left(f + \frac{2}{f}\frac{\partial M^*}{R\partial R}\right)^2 = 0, \tag{2.97a}$$

with the set of associated boundary conditions

$$c_p\frac{\partial M^*}{\partial S} - M^* + \frac{1}{2f^2}\left(\frac{\partial M^*}{\partial R}\right) = 0 \quad \text{at } S = S_B, \tag{2.97b}$$

$$\frac{\partial M^*}{\partial S} = T_T \qquad \text{at } S = S_T, \tag{2.97c}$$

$$\frac{\partial M^*}{\partial R} = 0 \qquad \text{at } R = 0, \tag{2.97d}$$

$$M^* = \psi(S) \qquad \text{at } R = R_L, \tag{2.97e}$$

where $\psi(S)$ has been defined previously. Together (2.97a)–(2.97e) constitute a 2-D inversion operator which is the s-version of the invertibility principle presented in Schubert and Alworth (1987) [Eqs. (29a)- (29e)].

Chapter 3

THE MIXED-BALANCE THEORY ON THE SPHERE

In this chapter, a three dimentional mixed geostrophic and gradient (zonal) balanced theory on the sphere will be derived. This theory is analogous to the three dimensional Eliassen balanced vortex theory and is a generalization of the f-plane theory to full spherical geometry. Therefore the section plan in this chapter will proceed in a similar fashion to that of the last chapter. In section 3.1, we review some of the basics of the primitive equations in spherical coordinates. Beginning with section 3.2 we will derive a mixed-balance system and the conservation laws associated with the approximated system. Section 3.3 deals with the transformation of the balanced system by a set of combined geostrophic, potential latitude and entropy coordinates. In section 3.4, the vorticity, potential vorticity and potential pseudodensity equations associated with the new balanced system will be derived. The last equation turns out to be the fundamental predictive equation for the balanced system. The invertibility principle is discussed in section 3.5. In the final section, section 3.6, we will show how the zonally symmetric balanced theory (Hack *et al.*, 1989; Schubert *et al.*, 1991) falls into our generalized theory as a two dimensional special case.

3.1 The primitive equations

Let us begin with the Newtonian momentum equation in its vector form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\nabla \times \mathbf{u}) \times \mathbf{u} + f\mathbf{k} \times \mathbf{u} + \tfrac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) + \frac{1}{\rho}\nabla p + \mathbf{g} = \mathbf{F}, \tag{3.1}$$

where $\mathbf{u} = (u, v, w)$ is the three dimensional velocity, $\mathbf{k}$ the unit vector in the vertical direction, and F the frictional and other external body forces. Also note that we have adopted the traditional approximation (Phillips, 1966) so that $f = 2\Omega \sin\phi$ is twice the vertical component of the earth's rotation, which is a function of latitude.

Since we are attempting to develop a theory that can describe the large scale fluid motions on the globe, the set of spherical coordinates (λ, ϕ, z) with λ being the longitude, ϕ the latitude and z the vertical altitude, is most appropriate for such a discussion. The gradient vector in this coordinate system is

$$\nabla = \left(\frac{\partial}{a\cos\phi\partial\lambda}, \frac{\partial}{a\partial\phi}, \frac{\partial}{\partial z}\right). \tag{3.2}$$

Thus, (3.1) can be decomposed into three component equations in the spherical coordinate system

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{a\cos\phi\partial\lambda} + v\frac{\partial u}{a\partial\phi} + w\frac{\partial u}{\partial z} - \left(2\Omega\sin\phi + \frac{u\tan\phi}{a}\right)v + \frac{1}{\rho}\frac{\partial p}{a\cos\phi\partial\lambda} = F_\lambda, \tag{3.3}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{a\cos\phi\partial\lambda} + v\frac{\partial v}{a\partial\phi} + w\frac{\partial v}{\partial z} + \left(2\Omega\sin\phi + \frac{u\tan\phi}{a}\right)u + \frac{1}{\rho}\frac{\partial p}{a\partial\phi} = F_\phi, \tag{3.4}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{a\cos\phi\partial\lambda} + v\frac{\partial w}{a\partial\phi} + w\frac{\partial w}{\partial z} + g + \frac{1}{\rho}\frac{\partial p}{\partial z} = F_z, \tag{3.5}$$

where now (u, v, w) stand for the zonal, meridional and vertical winds respectively.

Following the discussions in Chapter 2, we now introduce the hydrostatic approximation into our dynamic system, and choose $s = c_p\ln(\theta/\theta_0)$, the specific entropy, as the vertical coordinate. This, of course, leads to the set of primitive equations expressed in the spherical and entropy coordinate system (λ, ϕ, s),

$$\frac{Du}{Dt} - \left(2\Omega\sin\phi + \frac{u\tan\phi}{a}\right)v + \frac{\partial M}{a\cos\phi\partial\lambda} = F_\lambda, \tag{3.6}$$

$$\frac{Dv}{Dt} + \left(2\Omega\sin\phi + \frac{u\tan\phi}{a}\right)u + \frac{\partial M}{a\partial\phi} = F_\phi, \tag{3.7}$$

$$\frac{\partial M}{\partial s} = T, \tag{3.8}$$

$$\frac{D\sigma}{Dt} + \sigma\left(\frac{\partial u}{a\cos\phi\partial\lambda} + \frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial\dot{s}}{\partial s}\right) = 0, \tag{3.9}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{a\cos\phi\partial\lambda} + v\frac{\partial}{a\partial\phi} + \dot{s}\frac{\partial}{\partial s}, \tag{3.10}$$

is the total derivative, $M = c_pT + gz$ the Montgomery potential, $\sigma = -\partial p/\partial s$ the pseudodensity in s-space. Note that (3.9), the continuity equation, has been added to the system. Since entropy is used as the vertical coordinate, the thermodynamic equation is implicit in the coordinate system. While the frictional force and diabatic heating are either specified or given by some parameterization scheme, together with the equation of state, (3.6)–(3.9) form a closed system, namely the five equations are to be used to solve for the five unknowns u, v, M, T and σ with the independent variables λ, ϕ and s, given the proper initial and boundary conditions.

3.1.1 The conservation principles

We next discuss the conservation principles associated with the primitive equations in spherical coordinates. The purpose of this discussion is to set up a parallel comparison with the balanced system that we are going to develop in the next section.

a. Angular momentum conservation

The absolute angular momentum in spherical coordinates is commonly defined as

$$m = a\cos\phi(u + \Omega a\cos\phi). \tag{3.11}$$

On taking the material derivative of (3.11) and using (3.6), the absolute angular momentum principle is obtained in the form,

$$\frac{Dm}{Dt} + \frac{\partial M}{\partial \lambda} = a\cos\phi F_\lambda. \tag{3.12}$$

When a zonally-symmetric flow is considered, the absolute angular momentum is a materially conserved quantity. This equation can be written in a flux form

$$\frac{\partial(\sigma m)}{\partial t} + \frac{\partial(\sigma u m)}{a\cos\phi\partial\lambda} + \frac{\partial(\sigma v m\cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial(\sigma\dot{s}m)}{\partial s} + \sigma\frac{\partial M}{\partial\lambda} = a\cos\phi\sigma F_\lambda, \tag{3.13}$$

by using the continuity equation (3.9). We can integrate (3.13) zonally at a certain latitude by imposing the periodic boundary condition from λ to $\lambda + 2\pi$. Since we are using entropy as the vertical coordinate, the problem arises when we integrate the same equation vertically in the case where isentropes terminate at the earth's surface. If such a case occurs, we adopt the massless layer approach proposed by Lorenz (1955) and used by many others in many different contexts, e.g., Bretherton (1966), Andrews (1983), Hoskins *et al.* (1985), Magnusdottir and Schubert (1990, 1991) and Fulton and Schubert (1991).

We assume that the isentropic surfaces, when terminated at the earth's surface, continuously extend under the surface with pressure equal to surface pressure. There is no mass trapped between the isentropes that run under the earth's surface. Let us regard the bottom isentropic surface s_B as the largest value of s which remains everywhere below the earth's surface. We further assume that $\dot{s}$ vanishes at both the top s_T and the bottom s_B boundaries. Thus (3.13) becomes

$$\frac{\partial}{\partial t}\iint m\sigma a\cos\phi d\lambda ds + \frac{\partial}{a\partial\phi}\iint mv\sigma a\cos\phi d\lambda ds = -\iint\frac{\partial M}{\partial\lambda}\sigma a\cos\phi d\lambda ds$$
$$+\iint(a\cos\phi F_\lambda)\sigma a\cos\phi d\lambda ds, \tag{3.14}$$

after integration vertically and zonally. This equation states that the local time rate of change of vertically and zonally integrated absolute angular momentum is due to the meridional flux of the angular momentum across the latitude boundaries and the generation (or dissipation) forced by the pressure and frictional torques represented by the two terms on the right-hand-side of (3.14).

b. Energy conservation

The kinetic energy principle can be obtained by adding u times (3.6) and v times (3.7). In doing so, we obtain

$$\frac{DK}{Dt} + u\frac{\partial M}{a\cos\phi\partial\lambda} + v\frac{\partial M}{a\partial\phi} = uF_\lambda + vF_\phi, \tag{3.15}$$

where $K = \frac{1}{2}(u^2 + v^2)$ is the quasi-static form of kinetic energy. Combining this result with (3.9) we obtain

$$\frac{\partial(\sigma K)}{\partial t} + \frac{\partial(\sigma uK)}{a\cos\phi\partial\lambda} + \frac{\partial(\sigma vK\cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial(\sigma\dot{s}K)}{\partial s} + \sigma u\frac{\partial M}{a\cos\phi\partial\lambda} + \sigma v\frac{\partial M}{a\partial\phi} = \sigma(uF_\lambda + vF_\phi). \tag{3.16}$$

After manipulation of (3.16) using the continuity and hydrostatic equations, the mass-weighted kinetic energy equation can be written as

$$\frac{\partial}{\partial t}(\sigma K) + \frac{\partial}{a\cos\phi\partial\lambda}(\sigma u(K+gz)) + \frac{\partial}{a\cos\phi\partial\phi}(\sigma v(K+gz)\cos\phi)$$
$$+\frac{\partial}{\partial s}\left(\sigma\dot{s}(K+gz) - gz\frac{\partial p}{\partial t}\right) + \sigma\alpha\omega = \sigma(uF_\lambda + vF_\phi), \tag{3.17}$$

where α is the specific volume and $\omega = Dp/Dt$ is the vertical velocity in the p-coordinate. We see from (3.17) that the kinetic energy is locally changed through the total energy fluxes that cross the domain boundaries and the conversion of available potential energy within the model domain. External forces may dissipate kinetic energy by forcing the system to do work.

The thermodynamic energy equation can be written in a flux form by using the mass continuity equation (3.9). It yields

$$\frac{\partial}{\partial t}(\sigma c_p T)+\frac{\partial}{a\cos\phi\partial\lambda}(\sigma u c_p T)+\frac{\partial}{a\cos\phi\partial\phi}(\sigma v c_p T\cos\phi)+\frac{\partial}{\partial s}(\sigma\dot{s}c_p T)-\sigma\alpha\omega = \sigma Q, \quad (3.18)$$

where $Q = T\dot{s}$ is the diabatic heating. The addition of (3.17) and (3.18) results in the cancellation of the conversion term $\sigma\alpha\omega$ and leads to the total energy equation

$$\frac{\partial}{\partial t}(\sigma(K+c_pT))+\frac{\partial}{a\cos\phi\partial\lambda}(\sigma u(K+M))+\frac{\partial}{a\cos\phi\partial\phi}(\sigma v(K+M)\cos\phi)$$
$$+\frac{\partial}{\partial s}\left(\sigma\dot{s}(K+M)-gz\frac{\partial p}{\partial t}\right) = \sigma(uF_\lambda+vF_\phi+Q). \quad (3.19)$$

Here we clearly see how diabatic heating and external forces enter and affect the dynamic system as external physical processes.

Again we take the same view of the lower boundary as in the massless layer approach (Lorenz, 1955; Hoskins *et al.*, 1985; Schubert *et al.*, 1989; Magnusdottir and Schubert, 1990, 1991) when the isentropes intercept the earth's surface. Regarding the bottom isentropic surface s_B as the largest value of s which remains everywhere below the earth's surface, assuming that the top boundary is both an isentropic and isobaric surface, and assuming no topography and vanising $\dot{s}$ at the top and bottom, we can integrate the total energy equation over the whole model domain that is large enough so that any horizontal energy flux vanishes at the lateral boundaries. In doing so, we obtain

$$\frac{\partial}{\partial t}\iiint(K+c_pT)\sigma a^2\cos\phi d\lambda d\phi ds = \iiint(uF_\lambda+vF_\phi+Q)\sigma a^2\cos d\lambda d\phi ds. \quad (3.20)$$

Thus, when external and diabatic processes are neglected, the mass-integrated total energy is a globally conserved quantity.

c. The vorticity, potential vorticity and potential pseudodensity equations

To derive the vorticity equation, let us first write the momentum equations (3.6) and (3.7) in their rotational forms

$$\frac{\partial u}{\partial t} - \zeta v + \dot{s}\frac{\partial u}{\partial s} + \frac{\partial}{a\cos\phi\partial\lambda}[M + \tfrac{1}{2}(u^2+v^2)] = F_\lambda, \tag{3.21}$$

$$\frac{\partial v}{\partial t} + \zeta u + \dot{s}\frac{\partial v}{\partial s} + \frac{\partial}{a\partial\phi}[M + \tfrac{1}{2}(u^2+v^2)] = F_\phi, \tag{3.22}$$

where ζ is the vertical component of the absolute vorticity.

Taking $-\partial\cos\phi(\ )/a\cos\phi\partial\phi$ of (3.21) and $\partial(\ )/a\cos\phi\partial\lambda$ of (3.22), then adding the resulting equations, we obtain

$$\frac{D\zeta}{Dt} + \zeta\left(\frac{\partial u}{a\cos\phi\partial\lambda} + \frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi}\right) = \left(\xi\frac{\partial}{a\cos\phi\partial\lambda} + \eta\frac{\partial}{a\partial\phi}\right)\dot{s} + \frac{\partial F_\phi}{a\cos\phi\partial\lambda} - \frac{\partial(F_\lambda\cos\phi)}{a\cos\phi\partial\phi} \tag{3.23}$$

where

$$(\xi,\eta,\zeta) = \left(-\frac{\partial v}{\partial s}, \frac{\partial u}{\partial s}, 2\Omega\sin\phi + \frac{\partial v}{a\cos\phi\partial\lambda} - \frac{\partial(u\cos\phi)}{a\cos\phi\partial\phi}\right) \tag{3.24}$$

is the absolute vorticity. Equation (3.23) is the isentropic form of the vorticity equation, which indicates that the Lagrangian time rate of change of the vertical component of vorticity is related to the horizontal divergence or convergence of the flow field, the twisting of the horizontal vorticity into the vertical and the contributions from all the non-conservative forces. This equation can also be written in the flux form

$$\frac{\partial(\sigma P)}{\partial t} + \frac{\partial(u\sigma P - \xi\dot{s} - F_\phi)}{a\cos\phi\partial\lambda} + \frac{\partial((v\sigma P - \eta\dot{s} + F_\lambda)\cos\phi)}{a\cos\phi\partial\phi} = 0, \tag{3.25}$$

where $P = \zeta/\sigma$ is the potential vorticity. In deriving (3.25) from (3.23), we have used the fact that the divergence of the curl of any vector field identically vanishes, i.e.,

$$\frac{\partial\xi}{a\cos\phi\partial\lambda} + \frac{\partial(\eta\cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial\zeta}{\partial s} = 0. \tag{3.26}$$

Equation (3.25) is the Haynes-McIntyre theorem in spherical coordinates. It partially elucidates "IPV thinking", whereby mass transport can bring about PV anomalies due to diabatic processes. Two very fundamental concepts of potential vorticity have been identified from (3.25) by Haynes and McIntyre (1987, 1990). As stated in their theorem, due to the disappearance of the component of the flux vector normal to the isentropic surface, the Rossby-Ertel potential vorticity (i) "can not be transported across any isentropic surface," and (ii)"can neither be created nor destroyed within a layer bounded by two isentropic surfaces."

The Rossby-Ertel potential vorticity principle is obtained by eliminating the horizontal divergence between (3.23) and (3.9). The result is

$$\frac{DP}{Dt} = \frac{1}{\sigma}\left[\xi\frac{\partial \dot{s}}{a\cos\phi\partial\lambda} + \eta\frac{\partial \dot{s}}{a\partial\phi} + \zeta\frac{\partial \dot{s}}{\partial s} + \frac{\partial F_\phi}{a\cos\phi\partial\lambda} - \frac{\partial(F_\lambda\cos\phi)}{a\cos\phi\partial\phi}\right], \tag{3.27}$$

which indicates that the time evolution and the spatial distribution of potential vorticity can be calculated from the Lagrangian history of the diabatic heating and the other external processes. In the case of the advective process dominating diabatic and frictional processes, potential vorticity is conserved following the motion of a fluid element.

We next define the reciprocal of potential vorticity $\sigma^* = 2\Omega\sin\phi/P$ as the potential pseudodensity. It is so named because when substituting the definition of P, we have

$$\sigma^* = \frac{2\Omega\sin\phi}{\zeta}\sigma, \tag{3.28}$$

i.e., the potential pseudodensity is the density in the s-space (hence it is referred to as pseudodensity) that an air parcel would have if it were moved in such a way that its vertical component of absolute vorticity takes the value of the earth's vorticity. The substitution of this definition into (3.27) leads to the potential pseudodensity equation

$$\frac{D\sigma^*}{Dt} + \frac{\sigma^*}{\zeta}\left[\xi\frac{\partial \dot{s}}{a\cos\phi\partial\lambda} + \eta\frac{\partial \dot{s}}{a\partial\phi} + \zeta\frac{\partial \dot{s}}{\partial s} + \frac{\partial F_\phi}{a\cos\phi\partial\lambda} - \frac{\partial(F_\lambda\cos\phi)}{a\cos\phi\partial\phi}\right] = \frac{\sigma^*}{f}\beta v, \tag{3.29}$$

where

$$\beta = \frac{2\Omega\cos\phi}{a}. \tag{3.30}$$

Equation (3.29) should be compared with the potential pseudodensity equation on the f-plane, which was discussed in Chapter 2, i.e., (2.31), where there is no β-effect. Therefore, even for adiabatic and frictionless motion, the materially conservative property of potential pseudodensity on the sphere has to yield to β-forcing. Later we will show that for the balanced system that we are going to develop next, when transformed by a proper set of coordinates, (3.29) can be written in a simple flux form, due to the cancellation of this extra β-term.

3.2 The mixed balance equations on the sphere

In this section, we will extend the result obtained in Chapter 2 to the full spherical case. That is, we will develop a three dimensional, mixed geostrophic gradient balance theory with a latitude dependent Coriolis force in spherical coordinates.

The small Rossby number analysis conducted in Chapter 2 illuminates the idea that the inclusion of flow curvature will not destroy the adjustment process whereby the flow tries to evolve toward a balanced state. This implies that balanced dynamics should not have problems in some circumstances in dealing with the curved flow. We have successfully developed such a balanced theory for circular vortex flows in Chapter 2. For flows on the sphere with substantial curvature and asymmetric eddies (e.g., developing baroclinic waves, meandering jet streams or blocking patterns) is there a balanced theory that can do better than semigeostrophic theory where the curvature effect is completely missing? Following the methodology developed in Chapter 2, in the next subsection we will directly import in the geostrophic-gradient momentum approximation and show how the set of primitive equations can be approximated to a set of balanced equations, and then in the following subsection, we derive the set of conservation principles which ensure the approximated system is physically reasonable.

3.2.1 The governing equations with the geostrophic-gradient momentum approximation

In line with the geostrophic-gradient momentum approximation deduced from the small Rossby number analysis in Chapter 2, we approximate the set of primitive equations expressed in the spherical and entropy coordinate system (λ, ϕ, s) in the form:

$$\frac{Du_g}{Dt} - \left(2\Omega \sin\phi + \frac{u_g \tan\phi}{a}\right) v + \frac{\partial M}{a\cos\phi\partial\lambda} = 0, \tag{3.31}$$

$$\frac{Dv_g}{Dt} + \left(2\Omega \sin\phi + \frac{u_g \tan\phi}{a}\right) \frac{u}{\gamma} + \frac{\partial M}{\gamma a\partial\phi} = 0, \tag{3.32}$$

$$\frac{\partial M}{\partial s} = T, \tag{3.33}$$

$$\frac{D\sigma}{Dt} + \sigma\left(\frac{\partial u}{a\cos\phi\partial\lambda} + \frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial\dot{s}}{\partial s}\right) = 0, \tag{3.34}$$

where, for simplicity, the frictional force has been neglected. We would like to reemphasize that

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{a\cos\phi\partial\lambda} + v\frac{\partial}{a\partial\phi} + \dot{s}\frac{\partial}{\partial s}, \tag{3.35}$$

is the total derivative, and (u, v) the total zonal and meridional components of the velocity, while u_g the gradiently balanced zonal wind defined as

$$\left(2\Omega\sin\phi + \frac{u_g\tan\phi}{a}\right)u_g + \frac{\partial M}{a\partial\phi} = 0, \tag{3.36}$$

and v_g the geostrophically balanced meridional wind, i.e.,

$$fv_g = \frac{\partial M}{a\cos\Phi\partial\lambda}, \tag{3.37}$$

where the Coriolis parameter is redefined as

$$f = 2\Omega\sin\Phi. \tag{3.38}$$

Note that Φ, different from ϕ, is the potential latitude which will be defined later. Although the occurrence of two kinds of latitudes in the same dynamical system seems somewhat awkward, it is of benefit when coordinate transformations are introduced in the next section. The Coriolis force therefore, under such a definition, is evaluated at the transformed latitude, the potential latitude, rather than the physical latitude. Again as in Chapter 2, γ is a purely mathematically introduced quantity. Nevertheless, it does bear some physical meaning since it is written in the form

$$\gamma = \left(\frac{\cos\phi}{\cos\Phi}\right)\frac{2\Omega\sin\phi + u_g\tan\phi/a}{f}, \tag{3.39}$$

i.e., it measures the ratio of the combined planetary vorticity and relative curvature vorticity to the planetary vorticity itself. In accordance with this, we conjecture that when the curvature vorticity is small compared with the planetary vorticity, Φ approximately equals ϕ [see (3.49)], and $\gamma \approx 1$. Then the whole system reduces to the semigeostrophic equations. If we further replace (u_g, v_g) by their full counterparts (u, v), (3.31)–(3.32) revert to the primitive equations.

Since the total winds in the acceleration and curvature terms are approximated by geostrophic and gradient winds, we shall henceforth refer to (3.31)–(3.34) as the mixed geostrophic-gradient balance system. Together with the the equation of the state (Note that the thermodynamic equation is implicit in the coordinate system), (3.31)–(3.37) form a closed system, i.e., there are seven equations for the seven unknowns u, v, u_g, v_g, M, T and σ (the formula that defines the potential latitude will be given in the next section). Note that u_g and v_g are diagnosed from M field through (3.36) and (3.37), and M is related to σ (although the relation is not so obvious at this time, we will prove later in section 3.5 that in transformed space such a relation is the invertibility principle), while σ is predicted by (3.34). Therefore, (3.31) and (3.32) are no longer independent predictors. In this regard, only one prognostic equation is needed, i.e., the system has only one class of eigenfrequencies.

3.2.2 The conservation principles

The above mixed geostrophic and gradient balanced system preserves following physical laws:

a. The angular momentum conservation

We define the absolute gradient angular momentum as

$$m_g = a\cos\phi(u_g + \Omega a\cos\phi). \tag{3.40}$$

On taking the material derivative of (3.40) and using (3.31), the absolute angular momentum principle is easily obtained,

$$\frac{Dm_g}{Dt} + \frac{\partial M}{\partial \lambda} = 0. \tag{3.41}$$

For zonally symmetric flow, the second term in this equation disappears, resulting in conservation of absolute angular momentum. According to Hamilton's canonical equations, this implies that there must exist a corresponding cyclic coordinate that can be used to transform the momentum equation into a canonical form. In fact, as we will discuss later, $m_g = \Omega a^2\cos^2\Phi$ defines the potential latitude.

Similar to the previous discussion, we can write (3.41) in a flux form

$$\frac{\partial(\sigma m)}{\partial t} + \frac{\partial(\sigma u m)}{a\cos\phi\partial\lambda} + \frac{\partial(\sigma v m\cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial(\sigma\dot{s}m)}{\partial s} + \sigma\frac{\partial M}{\partial\lambda} = a\cos\phi\sigma F_\lambda, \tag{3.42}$$

by using the continuity equation. Considering cases when isentropes terminate at the ground, and applying the massless layer approach, the integration of (3.42) can be carried out both vertically and zonally to yield

$$\frac{\partial}{\partial t}\iint m_g \sigma a \cos\phi d\lambda ds + \frac{\partial}{a\partial\phi}\iint m_g v\sigma a \cos\phi d\lambda ds + \iint \frac{\partial M}{\partial\lambda}\sigma a \cos\phi d\lambda ds = 0, \quad (3.43)$$

which is the approximate form of the integrated angular momentum conservation principle (3.14) in the sense that the total angular momentum is now replaced by the gradient angular momentum.

b. The energy conservation

Adding u_g times (3.31) and v_g times (3.32), the kinetic energy equation is obtained accordingly

$$\frac{DK_g}{Dt} + u\frac{\partial M}{a\cos\phi\partial\lambda} + v\frac{\partial M}{a\partial\phi} = 0, \quad (3.44)$$

where $K_g = \frac{1}{2}(u_g^2 + v_g^2)$ is the combined geostrophic-gradient kinetic energy. Following the same procedures as we did for the primitive equation system, we can write this equation in a flux form

$$\frac{\partial}{\partial t}(\sigma K_g) + \frac{\partial}{a\cos\phi\partial\lambda}(\sigma u(K_g + gz)) + \frac{\partial}{a\cos\phi\partial\phi}(\sigma v(K_g + gz)\cos\phi)$$
$$+\frac{\partial}{\partial s}\left(\sigma\dot{s}(K_g + gz) - gz\frac{\partial p}{\partial t}\right) + \sigma\alpha\omega = 0. \quad (3.45)$$

Using the mass continuity equation (3.34), the thermodynamic energy equation can be written in the form:

$$\frac{\partial}{\partial t}(\sigma c_p T) + \frac{\partial}{a\cos\phi\partial\lambda}(\sigma u c_p T) + \frac{\partial}{a\cos\phi\partial\phi}(\sigma v c_p T\cos\phi) + \frac{\partial}{\partial s}(\sigma\dot{s}c_p T) - \sigma\alpha\omega = \sigma Q. \quad (3.46)$$

Again, the addition of (3.45) and (3.46) results in the cancellation of the conversion term $\sigma\alpha\omega$ and leads to a total energy equation:

$$\frac{\partial}{\partial t}(\sigma(K_g + c_p T)) + \frac{\partial}{a\cos\phi\partial\lambda}(\sigma u(K_g + M)) + \frac{\partial}{a\cos\phi\partial\phi}(\sigma v(K_g + M)\cos\phi)$$
$$+\frac{\partial}{\partial s}\left(\sigma\dot{s}(K_g + M) - gz\frac{\partial p}{\partial t}\right) = \sigma Q. \quad (3.47)$$

Assuming the top boundary is both an isentropic and isobaric surface, assuming no topography and vanising $\dot{s}$ at the top and bottom, we can integrate the total energy equation over a model domain that is large enough so that any horizontal energy flux vanishes at the lateral boundaries to give

$$\frac{\partial}{\partial t}\iiint (K_g + c_p T)\sigma a^2 \cos\phi d\lambda d\phi ds = \iiint Q\sigma a^2 \cos\phi d\lambda d\phi ds, \tag{3.48}$$

where we have taken into account of the cases when the lower isentropes terminate at the earth's surface by adopting the massless layer approach. In comparison of (3.48) with (3.20), we see that, except for the fact that the kinetic energy is evaluated with geostrophic and gradient winds, the governing equations (3.31)–(3.34) have a total energy conservation principle identical to the one which exists for the primitive equations.

We will delay the derivations of the vorticity equation and the potential vorticity conservation law in section 3.4 after the discussion of the coordinate transformation.

3.3 The combined geostrophic longitude, potential latitude and entropy coordinate transformation

Following the formalism of semigeostrophic theory (Hoskins, 1975; Schubert *et al.* 1989; Magnusdottir and Schubert, 1990; 1991), it is logical for us to seek a set of new coordinates which allow us to reformulate the problem (3.31)–(3.35) in the quasi-geostrophic space. Here the term " quasi-geostrophic space" possesses three meanings: (1) such a space is constituted by a set of quasi-geostrophic coordinates (Note: not strict geostrophic ones); (2) the governing equations reduce to the quasi-geostrophic formulation, i.e., one prognostic equation and one diagnostic equation; (3) the full advective winds become geostrophic (or gradient) and quasi-horizontal for adiabatic motion. As we have pointed out, due to the combined geostrophic and gradient momentum approximations, the two horizontal momentum equations have forfeited their status as independent predictors. Therefore, our goal is to transform (3.31) and (3.32) to their canonical forms, and to combine or replace these transformed equations so that they can ultimately form not only a closed but also a concise system. For flows on the sphere, these canonical equations are most likely to assume exactly the same forms as those derived by Magnusdottir and Schubert (1991) [Eq. (2.18) and (2.19)]. Since in this study we have included flow curvature, a different balance approximation has been used, and the question is just what transformed coordinates are appropriate. One may immediately realize that the questions have partially been answered by recalling the derivation of the angular momentum principle in which a cyclic coordinate is already implied. As a matter of fact, this cyclic coordinate has been discovered and used in the context of two dimensional, zonally symmetric flow on the sphere by Hack *et al.* (1989) and Schubert *et al.* (1990), where it was called "potential latitude". In accordance with the angular momentum principle, let us define this potential latitude as Φ to satisfy

$$a \cos^2 \Phi = \Omega a^2 \cos \phi (u_g + \Omega a \cos \phi). \tag{3.49}$$

From this definition we see that the potential latitude Φ is the latitude to which an air parcel must be moved (conserving its absolute angular momentum) in order for its relative angular momentum to vanish (Hack *et al.* 1989). It can also be written in the form

$$\sin \Phi = \sin \phi - \frac{u_g \cos \phi}{\Omega a (\sin \Phi + \sin \phi)}. \tag{3.50}$$

When ϕ is approximated by Φ in the second term on the right-hand-side of (3.50), this potential latitude formula reduces to the spherical geostrophic coordinate that has been used by Magnusdottir and Schubert (1991) in deriving the hemispherical semigeostrophic theory.

We next need to consider the other horizontal coordinate. Since we assumed that the meridional momentum is in geostrophic balance, it is natural to introduce the geostrophic longitude correspondingly, that is, the longitude fluid particles would have if they were moved with their geostrophic velocity at every instant. Mathematically, this coordinate can be written as

$$\Lambda = \lambda + \frac{v_g}{2\Omega a \sin \Phi \cos \Phi}. \tag{3.51}$$

Combining the potential latitude and geostrophic longitude coordinate as two new horizontal coordinates, (3.31)–(3.32) can, presumably, be transformed to their canonical forms, and the subsequent use of these canonical equations in the full advective operator makes the horizontal ageostrophic winds completely implicit. In consideration of the vertical advection, we again encounter the duality of using the isentropic coordinate in the vertical and the quasi-geostrophic coordinates in the horizontal. As suggested in the previous discussion in Chapter 2, we see that the combined use of the isentropic and quasi-geostrophic coordinates to construct a full three dimensional space is a feasible approach. Therefore we define $S = s$ and $\mathcal{T} = t$ as the new vertical and time coordinates, noting that $\partial/\partial s$ and $\partial/\partial t$ imply fixed r, ϕ while $\partial/\partial S$ and $\partial/\partial \mathcal{T}$ imply fixed R, Φ. With these newly defined coordinates, we can now proceed to transform our balanced system (3.31)–(3.34) from (λ, ϕ, s, t) space to $(\Lambda, \Phi, S, \mathcal{T})$ space. The derivative relations in the two spaces are given by

$$\frac{\partial}{\partial t} = \frac{\partial \Lambda}{\partial t}\frac{\partial}{\partial \Lambda} + \frac{\partial \Phi}{\partial t}\frac{\partial}{\partial \Phi} + \frac{\partial}{\partial \mathcal{T}}, \tag{3.52}$$

$$\frac{\partial}{\partial \lambda} = \frac{\partial \Lambda}{\partial \lambda}\frac{\partial}{\partial \Lambda} + \frac{\partial \Phi}{\partial \lambda}\frac{\partial}{\partial \Phi}, \tag{3.53}$$

$$\frac{\partial}{\partial \phi} = \frac{\partial \Lambda}{\partial \phi}\frac{\partial}{\partial \Lambda} + \frac{\partial \Phi}{\partial \phi}\frac{\partial}{\partial \Phi}, \tag{3.54}$$

$$\frac{\partial}{\partial s} = \frac{\partial \Lambda}{\partial s}\frac{\partial}{\partial \Lambda} + \frac{\partial \Phi}{\partial s}\frac{\partial}{\partial \Phi} + \frac{\partial}{\partial S}, \tag{3.55}$$

Applying this set to the Bernoulli function $M^* = M + \frac{1}{2}(u_g^2 + v_g^2)$, we can prove the following relations

$$\left(\frac{\partial M}{\partial \lambda}, \frac{\partial M}{\gamma \partial \phi}, \frac{\partial M}{\partial s}, \frac{\partial M}{\partial t}\right) = \left(\frac{\partial M^*}{\partial \Lambda}, \frac{\partial M^*}{\partial \Phi} - \frac{v_g^2(\cos^2\Phi - \sin^2\Phi)}{\sin\Phi\cos\Phi}, \frac{\partial M^*}{\partial S}, \frac{\partial M^*}{\partial \mathcal{T}}\right). \tag{3.56}$$

The transformation relations (3.52)–(3.55) also imply that the total derivative (3.35) can be written as

$$\frac{D}{Dt} = \frac{\partial}{\partial \mathcal{T}} + U\frac{\partial}{a\cos\Phi\partial\Lambda} + V\frac{\partial}{a\partial\Phi} + \dot{S}\frac{\partial}{\partial S}, \tag{3.57}$$

where

$$\left(U,\, V,\, \dot{S}\right) = \left(a\cos\Phi\frac{D\Lambda}{Dt},\, a\frac{D\Phi}{Dt},\, \frac{DS}{Dt}\right), \tag{3.58}$$

is the vector velocity in transformed space and $\dot{s} = \dot{S}$.

Finally with the aid of (3.56) we can now show that (3.31) and (3.32) take the canonical forms (see appendix A for detailed derivation)

$$2\Omega \sin \Phi a \frac{D\Phi}{Dt} = \frac{\partial M^*}{a \cos \Phi \partial \Lambda}, \tag{3.59}$$

$$-2\Omega \sin \Phi a \cos \Phi \frac{D\Lambda}{Dt} = \frac{\partial M^*}{a \partial \Phi}, \tag{3.60}$$

which are exactly the same results as those obtained by Magnusdottir and Schubert (1991). It is interesting to note that the horizontal advective winds in the Lagrangian time derivative in (3.57) are related to the Bernoulli function in such a way that they are formally in geostrophic balance. These advecting momenta are solely determined by geostrophic and gradient winds in physical space through (3.56), (3.36) and (3.37). Therefore, the major advantage of the transformation from (λ, ϕ, s, t) space to (Λ, Φ, S, T) space is that the two momentum equations are reduced to their canonical forms and substitutions of these equations result in the absence of ageostrophic advection in (3.57). In addition, for adiabatic flow the vertical advection does not appear in (3.57), so that the total advective operator becomes quasi-horizontal in such a coordinate space.

3.4 Vorticity, potential vorticity and potential pseudodensity equations

The conventional way to derive the vorticity equation is to take the curl of the momentum equation, namely, to take the cross derivatives of the horizontal momentum equations and then to combine them with the hydrostatic approximation. Now that the horizontal momentum equations for our mixed-balance, isentropic system have been transformed into their canonical forms (3.59) and (3.60), the simplest way to derive the vorticity equation associated with this system is to take the cross derivative of (3.59) and (3.60) and then to combine them in such a way as to form the total derivative of $2\Omega \sin \Phi \partial(\Lambda, \sin \Phi)/(\partial\lambda, \sin\phi)$, i.e., to form $\Lambda_\lambda[(3.59)\cos\Phi]_{\sin\phi} - (\sin\Phi)_{\sin\phi}[(3.60)/\cos\Phi]_\lambda$ $-\Lambda_{\sin\phi}[(3.59)\cos\Phi]_\lambda + (\sin\Phi)_\lambda[(3.60)/\cos\Phi]_{\sin\phi}$. In doing so, we obtain (the detailed derivation is given in appendix B)

$$\frac{D\zeta_g}{Dt} + \zeta_g \left(\frac{\partial u}{a \cos\phi \partial \lambda} + \frac{\partial (v \cos\phi)}{a \cos \phi \partial \phi} \right) - \left(\xi_g \frac{\partial}{\partial \lambda} + \eta_g \frac{\partial}{\partial \sin \phi} \right) \dot{s} = 0, \tag{3.61}$$

where

$$(\xi_g, \eta_g, \zeta_g) = f\left(\frac{\partial(\Lambda, \sin\Phi)}{\partial(\sin\phi, s)}, \frac{\partial(\Lambda, \sin\Phi)}{\partial(s, \lambda)}, \frac{\partial(\Lambda, \sin\Phi)}{\partial(\lambda, \sin\phi)}\right) \tag{3.62}$$

is the vorticity vector associated with the geostrophic and gradient winds. If we follow a traditional definition of potential vorticity (the Rossby-Ertel type) in the isentropic coordinate form

$$P_g = \frac{\zeta_g}{\sigma}, \tag{3.63}$$

and note the vector identity

$$\frac{\partial \xi_g}{a\cos\phi\partial\lambda} + \frac{\partial(\eta_g \cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial\zeta_g}{\partial s} = 0, \tag{3.64}$$

(3.61) can be written in the flux form

$$\frac{\partial(\sigma P_g)}{\partial t} + \frac{\partial(u\sigma P_g - \xi_g \dot{s})}{a\cos\phi\partial\lambda} + \frac{\partial((v\sigma P_g - \eta_g \dot{s})\cos\phi)}{a\cos\phi\partial\phi} = 0, \tag{3.65}$$

which again leads to the Haynes-McIntyre theorem.

Before we derive the potential vorticity principle, we would like to show the following useful relations. From (3.53)–(3.55) we have

$$\frac{\partial(\Lambda, \sin\Phi)}{\partial(\lambda, \sin\phi)}\frac{\partial}{\partial\Lambda} = \frac{\partial\sin\Phi}{\partial\sin\phi}\frac{\partial}{\partial\lambda} - \frac{\partial\sin\Phi}{\partial\lambda}\frac{\partial}{\partial\sin\phi}, \tag{3.66}$$

$$\frac{\partial(\Lambda, \sin\Phi)}{\partial(\lambda, \sin\phi)}\frac{\partial}{\partial\sin\Phi} = -\frac{\partial\Lambda}{\partial\sin\phi}\frac{\partial}{\partial\lambda} + \frac{\partial\Lambda}{\partial\lambda}\frac{\partial}{\partial\sin\phi}. \tag{3.67}$$

In making use of (3.66) and (3.67) in (3.55) we can prove that

$$\xi_g\frac{\partial}{\partial\lambda} + \eta_g\frac{\partial}{\partial\sin\phi} + \zeta_g\frac{\partial}{\partial s} = \zeta_g\frac{\partial}{\partial S}. \tag{3.68}$$

This relation shows that $\partial/\partial S$ is actually the derivative along the vorticity vector, and that is why we sometimes refer to $(\Lambda, \Phi, S, \mathcal{T})$ as "vortex coordinates".

The potential vorticity equation is derived by combining the vorticity equation (3.61) and the continuity equation (3.34). The result is

$$\sigma\frac{DP_g}{Dt} = \left(\xi_g\frac{\partial}{\partial\lambda} + \eta_g\frac{\partial}{\partial\sin\phi} + \zeta_g\frac{\partial}{\partial s}\right)\dot{s} = \zeta_g\frac{\partial\dot{s}}{\partial S}, \tag{3.69}$$

which gives exactly the same dynamical statement we obtained from the primitive system, i.e., potential vorticity is a materially conserved quantity when the advective process dominates diabatic and any other dissipative processes (here the external forces have already been neglected).

We next derive an equation for the inverse of potential vorticity, i.e., the potential pseudodensity equation. In an analogous way to the discussion of the primitive equations, let us define the potential pseudodensity as

$$\sigma_g^* = \frac{2\Omega \sin \Phi}{\zeta_g} \sigma, \tag{3.70}$$

so that the potential vorticity P_g and the potential pseudodensity σ_g^* are related by $P_g \sigma_g^* = 2\Omega \sin \Phi$. On substituting this definition into (3.69), we arrive at

$$\frac{D\sigma_g^*}{Dt} + \sigma_g^* \frac{\partial \dot{s}}{\partial S} = \frac{\beta V \sigma_g^*}{f}, \tag{3.71}$$

where

$$\beta = \frac{2\Omega \cos \Phi}{a}, \tag{3.72}$$

which should be comparable with that in the primitive case. The β-effect serves an additional forcing for potential pseudodensity apart from diabatic heating. By using (3.57)–(3.58), we can write the potential pseudodensity equation in a flux form

$$\frac{\partial \sigma_g^*}{\partial T} + \frac{\partial (U \sigma_g^*)}{a \cos \Phi \partial \Lambda} + \frac{\partial (V \sigma_g^* \cos \Phi)}{a \cos \Phi \partial \Phi} + \frac{\partial (\dot{S} \sigma_g^*)}{\partial S} = 0, \tag{3.73}$$

where U and V in the horizontal flux terms are given in (3.58), and they are related to a single variable M^* through (3.59) and (3.60). Note that $\dot{S} = \dot{s}$ is the diabatic heating which is either specified or given by some kind of parameterization. Thus, the integration of (3.73) forward in time requires only the initial σ_g^* field and the history of the diabatic heating provided that M^* is somehow obtainable, which is the topic of the next section.

3.5 Invertibility principle

In order to complete the predictive cycle, we shall next search for a diagnostic equation which can be used to invert the predicted σ_g^* to the basic diagnostic variable M^*. We begin with the definition of σ_g^* (3.70), which can be written as

$$\frac{\partial(\lambda, \sin\phi)}{\partial(\Lambda, \sin\Phi)}\frac{\partial p}{\partial s} + \sigma^* = 0, \tag{3.74}$$

by noting (3.62). Applying (3.55) to λ and $\sin\phi$ respectively, and combining the two resultant equations in such ways as to yield

$$\frac{\partial\Lambda}{\partial s}\frac{\partial\lambda}{\partial\Lambda}\frac{\partial\sin\phi}{\partial\sin\Phi} - \frac{\partial\Lambda}{\partial s}\frac{\partial\sin\phi}{\partial\Lambda}\frac{\partial\lambda}{\partial\sin\Phi} = \frac{\partial\sin\phi}{\partial S}\frac{\partial\lambda}{\partial\sin\Phi} - \frac{\partial\lambda}{\partial S}\frac{\partial\sin\phi}{\partial\sin\Phi}, \tag{3.75}$$

and

$$\frac{\partial\sin\Phi}{\partial s}\frac{\partial\lambda}{\partial\sin\Phi}\frac{\partial\sin\phi}{\partial\Lambda} - \frac{\partial\sin\Phi}{\partial s}\frac{\partial\sin\phi}{\partial\sin\Phi}\frac{\partial\lambda}{\partial\Lambda} = \frac{\partial\sin\phi}{\partial S}\frac{\partial\lambda}{\partial\Lambda} - \frac{\partial\lambda}{\partial S}\frac{\partial\sin\phi}{\partial\Lambda}, \tag{3.76}$$

we then substitute (3.75) and (3.76) in the expansion of the first term in (3.74) to obtain

$$\begin{aligned}\frac{\partial(\lambda, \sin\phi)}{\partial(\Lambda, \sin\Phi)}\frac{\partial p}{\partial s} &= \frac{\partial\sin\phi}{\partial S}\frac{\partial\lambda}{\partial\sin\Phi}\frac{\partial p}{\partial\Lambda} - \frac{\partial\lambda}{\partial S}\frac{\partial\sin\phi}{\partial\sin\Phi}\frac{\partial p}{\partial\Lambda} + \frac{\partial\sin\phi}{\partial S}\frac{\partial\lambda}{\partial\Lambda}\frac{\partial p}{\partial\sin\Phi} \\ &\quad - \frac{\partial\lambda}{\partial S}\frac{\partial\sin\phi}{\partial\Lambda}\frac{\partial p}{\partial\sin\Phi} + \frac{\partial\lambda}{\partial\Lambda}\frac{\partial\sin\phi}{\partial\sin\Phi}\frac{\partial p}{\partial S} - \frac{\partial\lambda}{\partial\sin\Phi}\frac{\partial\sin\phi}{\partial\Lambda}\frac{\partial p}{\partial S} \\ &= \frac{\partial(\lambda, \sin\phi, p)}{\partial(\Lambda, \sin\Phi, S)}.\end{aligned}$$

Thus, we can write (3.74) in the Jacobian form

$$\frac{\partial(\lambda, \sin\phi, p)}{\partial(\Lambda, \sin\Phi, S)} + \sigma_g^* = 0, \tag{3.77}$$

where now λ, $\sin\phi$, p and σ^* are expressed as dependent variables in $(\Lambda, \Phi, S, \mathcal{T})$ space. We notice that the additional term in the second entry of (3.56) presents a small correction to the mixed geostrophically and zonally balanced flow on the sphere. This can be easily seen by comparing this term to the term on the left hand side of (3.56), i.e.,

$$\frac{\left|v_g^2(\cos^2\Phi - \sin^2\Phi)/\sin\Phi\cos\Phi\right|}{|\partial M/\gamma\partial\phi|} = \left|\frac{2v_g^2\cos\phi\cos 2\Phi}{fau_g\cos\Phi\sin 2\Phi}\right| \le \left|\frac{v_g^2\cos 2\Phi}{\Omega au_g\sin\Phi\sin 2\Phi}\right|$$

for westerly zonal flow. If we consider the typical magnitude of the geostrophic wind $u_g \sim v_g \sim 10ms^{-1}$, this ratio is about 1/40 at 30°N, 1/75 at 60°N. These values are even overestimated somewhat because, in a normal physical situation, the zonal wind dominates the meridional wind by as much as one order of magnitude, specially at middle and upper atmospheric levels. For this reason the small additional term can be dropped, which results in the geostrophic, gradient and hydrostatic relations in transformed space taking the forms

$$(fv_g, -fu_g, T) = \left(\frac{\partial M^*}{a\cos\Phi\partial\Lambda}, \frac{\cos\phi}{\cos\Phi}\frac{\partial M^*}{a\partial\Phi}, \frac{\partial M^*}{\partial S}\right). \tag{3.78}$$

Using this set of relations in the geostrophic longitude (3.51), potential latitude (3.49) and the ideal gas law, we can write λ, $\sin\phi$ and p all in terms of M^* as

$$\lambda = \Lambda - \frac{\partial M^*}{f^2a^2\cos^2\Phi\partial\Lambda}, \tag{3.79}$$

$$\sin\phi = \sin\Phi - \frac{\cos\Phi\partial M^*}{f^2a^2\partial\Phi}, \tag{3.80}$$

$$p = \rho \mathsf{R}\frac{\partial M^*}{\partial S}, \tag{3.81}$$

where R is the gas constant. We have also used the assumption in (3.80) that the physical latitude does not differ substantially from the potential latitude This assumption, however, is not indispensable to derive the invertibility principle. On substituting these relations in (3.77), we obtain

$$\frac{1}{f^4}\begin{vmatrix} \frac{\partial^2 M^*}{a^2\cos^2\Phi\partial\Lambda^2} - f^2 & f^2\frac{\partial}{a\partial\Phi}\left(\frac{1}{f^2}\frac{\partial M^*}{a\cos^2\Phi\partial\Lambda}\right) & \frac{\partial^2 M^*}{a\cos^2\Phi\partial\Lambda\partial S} \\ \frac{\partial^2\partial M^*}{a^2\partial\Lambda\partial\Phi} & f^2\frac{\partial}{a\cos\Phi\partial\Phi}\left(\frac{\cos\Phi}{f^2}\frac{\partial^2 M^*}{a\partial\Phi^2}\right) - f^2 & \frac{\partial^2\partial M^*}{a^2\partial\Phi\partial S} \\ \frac{\partial}{\partial\Lambda}\left(\rho\mathsf{R}\frac{\partial M^*}{\partial S}\right) & \frac{\partial}{\partial\Phi}\left(\rho\mathsf{R}\frac{\partial M^*}{\partial S}\right) & \frac{\partial}{\partial S}\left(\rho\mathsf{R}\frac{\partial M^*}{\partial S}\right) \end{vmatrix} + \sigma^* = 0. \tag{3.82a}$$

This is the basic diagnostic equation that we are seeking; it determines M^* from known σ_g^* at each time step. Again we arrive at a three dimensional generalized Monge-Ampère equation. The uniqueness of the solution to this boundary value problem can be found in Pogorelov (1964). When we expand the determinant in this equation, we obtain a three dimensional nonlinear elliptic type of equation. Therefore we next discuss the boundary conditions that go along with this elliptic differential equation.

To integrate this three dimensional, second order partial differential equation, six boundary conditions are required. The vertical boundary conditions can be derived in a similar fashion as that in Schubert *et al.* (1989). By neglecting the effects of topography and assuming that the lower boundary is the constant height surface $z = 0$ and the isentropic surface, we conclude that $M = c_p T$ at $S = S_B$. Written in terms of M^*, this lower boundary condition becomes

$$c_p \frac{\partial M^*}{\partial S} - M^* + \frac{1}{2f^2}\left[\left(\frac{\partial M^*}{a\cos\Phi\partial\Lambda}\right)^2 + \left(\frac{\partial M^*}{a\partial\Phi}\right)^2\right] = 0 \quad \text{at } S = S_B. \tag{3.82b}$$

Again we have assumed that $\phi \simeq \Phi$ in deriving (3.82b) for the sake of the computational convenience, although it is not necessary to do so. The upper boundary is assumed to be an isentropic and isobaric surface, hence it is also an isothermal one. Directly applying the hydrostatic equation to this surface, we can write the upper boundary condition for (3.82a) as

$$\frac{\partial M^*}{\partial S} = T_T \quad \text{at } S = S_T, \tag{3.82c}$$

where T_T is the constant temperature at the top layer of the model domain. This upper boundary condition implies that we neglect any interesting physical process occurring at the top of the model domain.

The periodic boundary condition can be assumed when the integration is taken along a latitude around the globe, i.e.,

$$M^*(\Lambda) = M^*(\Lambda + 2\pi) \tag{3.82d}$$

The meridional lateral boundary conditions should be chosen on the basis of particular application. For example, in studying baroclinic instability, a middle latitude channel may be considered. In such a channel bounded by two latitudes, we may assume that the meridional component of the geostrophic wind vanishes at the channel boundaries, which gives

$$\frac{\partial M^*}{a\cos\Phi\partial\Lambda} = 0 \quad \text{at } \Phi = \Phi_1, \Phi_2 \tag{3.82e,f}$$

Another example would be the atmospheric motions on hemisphere such as the Rossby wave dispersion problem, or the interactions between the tropical atmosphere and higher latitudes. In this case, the meridional boundary conditions can be imposed at the equator and one of the poles. At any rate, the foregoing dynamical system is complete.

One might question how the solution of this system will differ from the one given by semigeostrophic theory (Magnusdottir and Schubert, 1991) since the inversion operators for the two systems are identical. The subtle difference rests with the evaluation procedure from the known M^* field to the wind field. The semigeostrophic system arrives at its wind fields through the geostrophic balance relations, while in the present mixed-balance system one component of the wind is evaluated through the geostrophic relation and the other component through the zonal balance relation with curvature information retained [ref. (3.78)]. The reason that we obtained the identical inversion operator to that of semigeostrophic one was that we adopted the approximation $\phi \simeq \Phi$ in (3.80). One can also derive a more rigorous inversion operator without this approximation. The procedure will be very straightforward, and has been demostrated in Chapter 2.

• *Summary of the mixed geostrophic-zonal balance model on the sphere*

Thus far we have completed our derivation of a mathematical model which filters transient gravity waves and is able to describe three dimensional curved flows on the sphere. Like many other balanced theories, the model consists of two fundamental equations: one prognostic equation and one diagnostic equation. The predictive variable is the potential pseudodensity from which both the mass and wind fields are retrieved by using the invertibility principle. A summary of this balanced model is given in Table 3.1.

3.6 Zonally symmetric dynamics

In this final section we demonstrate that the zonally symmetric balanced theory proposed by Hack *et al.* (1989) and Schubert *et al.* (1991) is, in fact, a two dimensional special case of our mixed zonal-geostrophic balanced theory discussed in this chapter. This result suggests that the theory presented in this chapter is consistent with zonally symmetric theory.

We begin by imposing the zonal symmetry assumption in our generalized three dimensional model, i.e., letting all terms involving $\partial/\partial\lambda$ (or $\partial/\partial\Lambda$ in the transformed space) be zero, except for the term $\partial\Lambda/\partial\lambda$ being unity. In doing so, from (3.36), (3.37) and (3.32), we deduce that

$$u = u_g, \quad u_a = 0, \tag{3.83}$$

$$v = v_a, \quad v_g = 0, \tag{3.84}$$

which indicate that the meridional wind becomes purely ageostrophic, while the zonal wind is purely gradient. This is precisely the view used in zonally symmetric balanced theory. With the use of (3.83) and (3.84) the set of governing equations (3.31)–(3.35) reduce to

$$\frac{Du}{Dt} - \left(2\Omega \sin\phi + \frac{u \tan\phi}{a}\right) v = 0, \tag{3.85}$$

$$\left(2\Omega \sin\phi + \frac{u \tan\phi}{a}\right) u + \frac{\partial M}{a\partial\phi} = 0, \tag{3.86}$$

$$\frac{\partial M}{\partial s} = T, \tag{3.87}$$

$$\frac{D\sigma}{Dt} + \sigma\left(\frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial\dot{s}}{\partial s}\right) = 0, \tag{3.88}$$

where the total derivative now becomes

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v\frac{\partial}{a\partial\phi} + \dot{s}\frac{\partial}{\partial s}. \tag{3.89}$$

By comparing with (2.1)–(2.5) in Schubert *et al.* (1991), one will see that these are the s-coordinate versions of the two-dimensional, zonally symmetric balanced equations on the sphere.

In transformed space, again, let us impose that $\partial/\partial\Lambda = 0$ in the system of (T3.1)–(T3.6). The fundamental predictive equation, i.e., the potential pseudodensity equation (T3.1) reduces to the very simple form

$$\frac{\partial\sigma^*}{\partial T} + \frac{\partial(\dot{s}\sigma^*)}{\partial S} = 0, \tag{3.90}$$

which is identical to the potential pseudodensity equation obtained in the zonally symmetric balanced theory [see Eq. (3.9) in Schubert *et al.*, 1991]. This equation can be integrated analytically by using the method of characteristics, and the integration can be carried out to any time level if the diabatic heating is explicitly given. Then the diagnostic inversion operation can be performed at the desired time level. This has been successfully done for the potential vorticity modeling of the ITCZ and the Hadley circulation in Schubert *et al.* (1991).

The reduction of the 3×3 determinant to a 2×2 one is immediately seen upon the zonally symmetric condition being applied in (3.77)

$$\frac{\partial(\sin\phi, p)}{\partial(\sin\Phi, S)} + \sigma^* = 0, \tag{3.91}$$

or by expanding the Jacobian and taking (3.81) into account,

$$\frac{\partial(\sin\phi)}{\partial\sin\Phi}\frac{\partial}{\partial S}\left(\rho\mathrm{R}\frac{\partial M^*}{\partial S}\right) - \frac{\partial(\sin\phi)}{\partial S}\frac{\partial}{\partial\sin\Phi}\left(\rho\mathrm{R}\frac{\partial M^*}{\partial S}\right) + \sigma^* = 0. \tag{3.92a}$$

Also, let us consider the potential latitude equation in the form

$$\sin^2\Phi = \sin^2\phi - \frac{u_g\cos\phi}{\Omega a}. \tag{3.93}$$

By substituting the zonal balanced wind relation (3.78), this equation can be written as

$$2\Omega^2 a^2 \sin\Phi\left(\frac{\sin^2\phi - \sin^2\Phi}{1-\sin^2\phi}\right) + \frac{\partial M^*}{\partial\sin\Phi} = 0. \tag{3.92b}$$

(3.92a,b) can be regarded as a pair of equations for M^* and $\sin\phi$, which constitutes the invertibility principle for 2-D zonally symmetric flows. The boundary conditions associated with these two differential equations are given as follows. The upper boundary condition takes exactly the same form as the 3-D case

$$\frac{\partial M^*}{\partial S} = T_T \qquad \text{at } S = S_T. \tag{3.92c}$$

For the lower boundary, the constant height and isentropic surface gives

$$c_p\frac{\partial M^*}{\partial S} - M^* + \tfrac{1}{2}u_g^2 = 0, \tag{3.94}$$

which, by substituting in (3.78) and (3.93), leads to

$$c_p\frac{\partial M^*}{\partial S} - M^* + \frac{\Omega^2 a^2(\sin^2\phi - \sin^2\Phi)^2}{2(1-\sin^2\phi)} = 0 \qquad \text{at } S = S_B. \tag{3.92d}$$

Since there was no explicit assumption of geostrophic balance in the 2-D zonally simmetric model, we may integrate (3.92a,b) from pole to pole without difficulty. Assuming that there is no difference between ϕ and Φ at the south and north poles, the lateral boundary conditions can then be written as

$$\sin\phi = 1 \qquad \text{at } \sin\Phi = 1; \tag{3.92e}$$

$$\sin\phi = -1 \qquad \text{at } \sin\Phi = -1. \tag{3.92f}$$

Equations (3.92a)–(3.92f) reproduce the diagnostic system obtained by Schubert *et al.* (1991).

Table 3.1: Summary of the mixed-balance model on the sphere.

$$\frac{\partial \sigma_g^*}{\partial T} + \frac{\partial (U\sigma_g^*)}{a\cos\Phi\partial\Lambda} + \frac{\partial (V\sigma_g^*\cos\Phi)}{a\cos\Phi\partial\Phi} + \frac{\partial (\dot{S}\sigma_g^*)}{\partial S} = 0, \qquad \text{(T3.1)}$$

$$\frac{1}{f^4}\begin{vmatrix} \frac{\partial^2 M^*}{a^2\cos^2\Phi\partial\Lambda^2} - f^2 & f^2\frac{\partial}{a\partial\Phi}\left(\frac{1}{f^2}\frac{\partial M^*}{a\cos^2\Phi\partial\Lambda}\right) & \frac{\partial^2 M^*}{a\cos^2\Phi\partial\Lambda\partial S} \\ \frac{\partial^2\partial M^*}{a^2\partial\Lambda\partial\Phi} & f^2\frac{\partial}{a\cos\Phi\partial\Phi}\left(\frac{\cos\Phi}{f^2}\frac{\partial^2 M^*}{a\partial\Phi^2}\right) - f^2 & \frac{\partial^2\partial M^*}{a^2\partial\Phi\partial S} \\ \frac{\partial}{\partial\Lambda}\left(\rho\mathrm{R}\frac{\partial M^*}{\partial S}\right) & \frac{\partial}{\partial\Phi}\left(\rho\mathrm{R}\frac{\partial M^*}{\partial S}\right) & \frac{\partial}{\partial S}\left(\rho\mathrm{R}\frac{\partial M^*}{\partial S}\right) \end{vmatrix} + \sigma^* = 0, \qquad \text{(T3.2)}$$

$$c_p\frac{\partial M^*}{\partial S} - M^* + \frac{1}{2f^2}\left[\left(\frac{\partial M^*}{a\cos\Phi\partial\Lambda}\right)^2 + \left(\frac{\partial M^*}{a\partial\Phi}\right)^2\right] = 0 \quad \text{at } S = S_B, \qquad \text{(T3.3)}$$

$$\frac{\partial M^*}{\partial S} = T_T \quad \text{at } S = S_T, \qquad \text{(T3.4)}$$

$$M^*(\Lambda) = M^*(\Lambda + 2\pi), \qquad \text{(T3.5)}$$

$$\frac{\partial M^*}{a\cos\Phi\partial\Lambda} = 0 \quad \text{at } \Phi = \Phi_1, \Phi_2. \qquad \text{(T3.6, T3.7)}$$

Chapter 4

FREE OSCILLATIONS IN BAROTROPIC CIRCULAR VORTICES

In this chapter, we investigate unforced wave motions in the context of reduced gravity waves. In particular, we would like to calculate the eigenvalues and eigenfunctions associated with given barotropic circular vortices. This classical problem had been studied long before the turn of this century, as Thomson (1879) first touched such a problem. The detailed analysis was also given in Lamb (1932, pp. 320–324). The analytical solutions from these studies indicated that for a laterally-confined, uniformly rotating fluid on an f-plane, the only possible transient modes are the clockwise and counterclockwise propagating inertia-gravity waves. This result, however, can be misleading when a more complicated basic flow pattern is considered, because surface gradient of the fluid, the so called "equivalent β-effect", may then be introduced in correspondence with the radial inhomogeneity in tangential flow field. Under such circumstance, the surface height gradient may act as a restoring mechanism so that some kind of frequency oscillations other than the inertia-gravity waves may be induced. This idea was also appreciated by Thomson (1880) when he studied wave oscillations in barotropic vortices by resorting the nondivergent barotropic dynamics with a basic state tangential flow which varies with radius.

Rossby (1939) reexamined this problem using a similar dynamical system in a cartesian frame but approximating the earth's geometry by a tangent plane, i.e., the β-plane. The linear solutions of this β-plane model presented the wave motions with low frequencies similar to those of Kelvin's (Thomson, 1880), and these waves are thenceforward named *Rossby waves.* More complete normal mode studies by using primitive equations were conducted by Matsuno (1966) and by Longuet-Higgins (1968) in the context of wave motions in the equatorial area and the planetary waves in Laplace's tidal equations, respectively. These studies all used a set of shallow water primitive equations linearized about a basic state of rest with the β-plane approximation or the full spherical geometry. Therefore, Rossby waves are presented as the allowable eigensolutions, which would otherwise be absent under the f-plane approximation. Similar study of planetary waves on β-plane with vertical structure of the atmosphere was found in Lindzen (1967).

Insofar as the problem was originally posed, i.e., the problem of wave motions in barotropic vortices on an f-plane with nonresting basic tangential flows, there have been not any, to our knowledge, complete calculations of eigenvalues and eigenfunctions available. This type of research is necessary in the sense that it may not only provide the fundamental understanding of wave dynamics in vortices, vortex breakdown, but it may also, by combining the stability analysis, make direct applications to the studies of certain weather phenomena, for example, the spiral-banded clouds in hurricanes, the comma clouds associated with extratropical cyclones, tornadoes and their parental vortex circulations such as hurricanes, supercells.

The plan for this chapter is as follows. In section 4.1, we analytically solve the set of shallow water primitive equations linerized about a basic state of rest. Using these analytically obtained eigensolutions, we illustrate that in barotropic vortices on an f-plane, if the basic state is at rest, the main wave motions are confined to inertia-gravity types. There are, however, Rossby type of modes implicit in the system, but they are in an inactive state. To activate these Rossby waves in such vorticies, a nonresting basic flow has to be considered. In section 4.2, we thus first discuss one type of nonresting basic flows: Rankine's combined vortex, and then following Kelvin's approach, we seek an analytical solution of the nondivergent barotropic system with Rankine's vortex as the basic state. After all these analytical treatments, in section 4.3 we finally conduct a numerical calculation using the shallow water primitive equation model. The numerical solutions are compared with the analytical ones.

4.1 The wave motions in a barotropic vortex with a basic state of rest

The dynamical system that is suitable to describle barotropic vortices is the set of shallow water equations in polar coordinates (r, ϕ), which may be written,

$$\frac{Du}{\partial t} - \left(f + \frac{v}{r}\right) v + g\frac{\partial h}{\partial r} = 0, \tag{4.1}$$

$$\frac{Dv}{\partial t} + \left(f + \frac{v}{r}\right) u + g\frac{\partial h}{r\partial \phi} = 0, \tag{4.2}$$

$$\frac{Dh}{Dt} + h\left(\frac{\partial(ru)}{r\partial r} + \frac{\partial v}{r\partial \phi}\right) = 0, \tag{4.3}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + v\frac{\partial}{r\partial \phi}, \tag{4.4}$$

and other notations are standard. We now consider a uniformly rotating vortex by which we mean a sheet of circular fluid bounded by a given radius a, and rotating about a vertical axis with a constant angular velocity f, where $f = 2\Omega \sin \phi_0$, the same as that of earth's at a given latitude ϕ_0. The tangential velocity of the fluid is everywhere uniform. To an observer in a relative frame rotating with the earth, this circular vortex represents a resting basic state with a constant height surface $(\bar{u}, \bar{v}, \bar{h}) = (0, 0, H)$. Linearizing (4.1)–(4.4) about this basic state, we may obtain

$$\frac{\partial u'}{\partial t} - fv' + g\frac{\partial h'}{\partial r} = 0, \tag{4.5}$$

$$\frac{\partial v'}{\partial t} + fu' + g\frac{\partial h'}{r\partial \phi} = 0, \tag{4.6}$$

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial(ru')}{r\partial r} + \frac{\partial v'}{r\partial \phi}\right) = 0, \tag{4.7}$$

where the prime quantities are the deviations from the basic state of rest. We assume that the perturbations have the following wave structures

$$\begin{pmatrix} u' \\ v' \\ h' \end{pmatrix} = \begin{pmatrix} \hat{u}(r) \\ \hat{v}(r) \\ \hat{h}(r) \end{pmatrix} e^{i(s\phi - \sigma t)}. \tag{4.8}$$

where σ denotes the frequency of wave perturbations, s the azimuthal wavenumber. Substituting this expression into (4.5)–(4.7), we may have the following linear system

$$\begin{pmatrix} -i\sigma & -f & g\frac{\partial}{\partial r} \\ f & -i\sigma & is\frac{g}{r} \\ H\frac{\partial r(\,)}{r\partial r} & is\frac{H}{r} & -i\sigma \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{pmatrix} = 0. \tag{4.9}$$

In order to solve this linear system, one can combine the three equations with σ as an eigenvalue to be determined. There sould be three roots for eigenvalue σ, one of which is $\sigma = 0$. For this root, the eigensystem reduces to

$$f\hat{v} = g\frac{\partial \hat{h}}{\partial r}, \tag{4.10}$$

$$-f\hat{u} = is\frac{g\hat{h}}{r}, \tag{4.11}$$

$$\frac{\partial(r\hat{u})}{r\partial r} + is\frac{\hat{v}}{r} = 0. \tag{4.12}$$

The first two equations give the geostrophic balance relations, and the third one is the nondivergent condition. Therefore, we conclude that $\sigma = 0$ represents the stationary geostrophic modes in the eigensystem (4.9). The eigenfunctions associated with these geostrophic modes of zero-frequency can be found by solving the linear system (4.10)–(4.12), which has the sole trivial solution of $(u', v', gh') = (0, 0, 0)$ for any nonzero wavenumbers, and has infinite families of solutions for $s = 0$ case (axisymmetric case), i.e.,

$$\begin{pmatrix} u' \\ v' \\ gh' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{g}{f}\frac{\partial \hat{h}}{\partial r} \\ g\hat{h} \end{pmatrix}, \tag{4.13}$$

where $\hat{h}$ can be any arbitrary function of r. This result indicates that for barotropic vortices on the f-plane with no relative flows to the earth's rotation, the Rossby-like modes manifest themselves as steady, axisymmetric swirling flows in geostrophic balance. These stationary geostrophic modes had been overlooked by Thomson (1879) and Lamb (1932).

When σ is non-trivial, (4.9) can be solved systematically as an eigenvalue problem. Let us combine the three linear equations into one, and solve, say, for $\hat{h}$ field, which gives

$$\frac{d^2\hat{h}}{dr^2} + \frac{1}{r}\frac{d\hat{h}}{dr} + \left(\kappa^2 - \frac{s^2}{r^2}\right)\hat{h} = 0, \tag{4.14}$$

where κ which assumes a set of discrete values satisfies the dispersion relation

$$\sigma^2 = f^2 + \kappa^2 gH. \tag{4.15}$$

The two roots of σ from this relation represent the clockwise and counterclockwise propagating inertia-gravity waves.

Equation (4.14) is the Bessel equation of order s. The outer boundary condition associated with this equation is given by requiring the vanishing of perturbation radial velocity u' at $r = a$. By simultaneously considering the first two entries of (4.9), this condition can be expressed as

$$r\frac{\partial \hat{h}}{\partial r} - \frac{fs}{\sigma}\hat{h} = 0. \tag{4.16}$$

Therefore, the solution of (4.14) corresponded to the finite condition in $\hat{h}$ at the vortex center $r = 0$ is found to be

$$\hat{h} = AJ_s(\kappa r). \tag{4.17}$$

where $J_s(\kappa r)$ is the Bessel function of order s, and A is a constant coefficient. In order to calculate the eigenvalues σ, let us rewrite (4.15) in the form

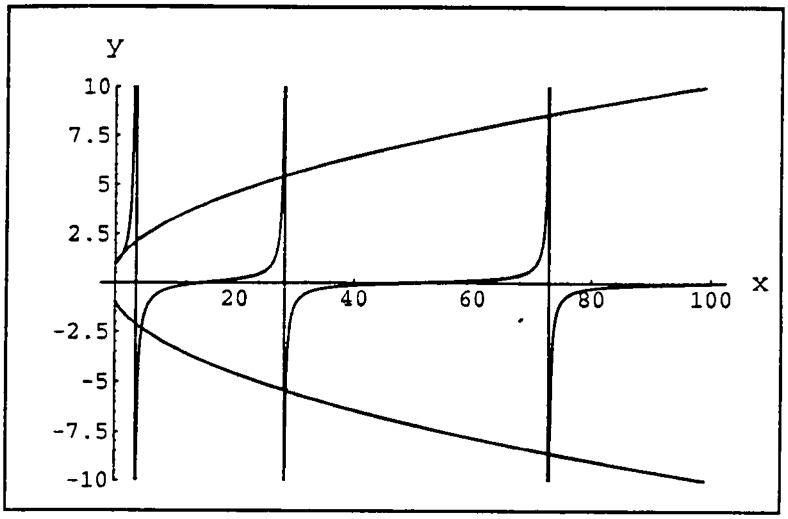
$$\frac{\sigma^2}{f^2} = 1 + \left(\frac{L_c}{a}\right)^2 \kappa^2 a^2, \tag{4.18}$$

where $L_c = \sqrt{gH}/f$ is the Rossby radius of deformation. Similarly, we can also rewrite (4.16) as

$$\frac{\sigma}{f} = \frac{sJ_s(\kappa a)}{\kappa a J_s'(\kappa a)} = \phi(\kappa^2 a^2). \tag{4.19}$$

where the primes denote the derivatives of the Bessel functions with respect to r. Denoting $y = \sigma/f$, and $x = \kappa^2 a^2$, (4.18) and (4.19) draw two curves, $y = \phi(x)$ and $y^2 = 1 + \alpha x$, in an (x, y) plane. The ordinates of intersections of these two curves give the set of normalized eigenvalues corresponded to the different radial modes (let us call them the n modes, following the convention). Figure 4.1 is shown the plot discussed above, where we have chosen that $\alpha = L_c^2/a^2 = gH/f^2a^2 \approx 1$, with $f = 2\Omega \sin 20°$, $H = 250$m and $a = 1000$km. The upper panel is for wavenumber $s = 1$, and the lower for $s = 5$. The eigenfrequencies that we found from these plots are listed in Table 4.1 for the first four n

(a)



(b)

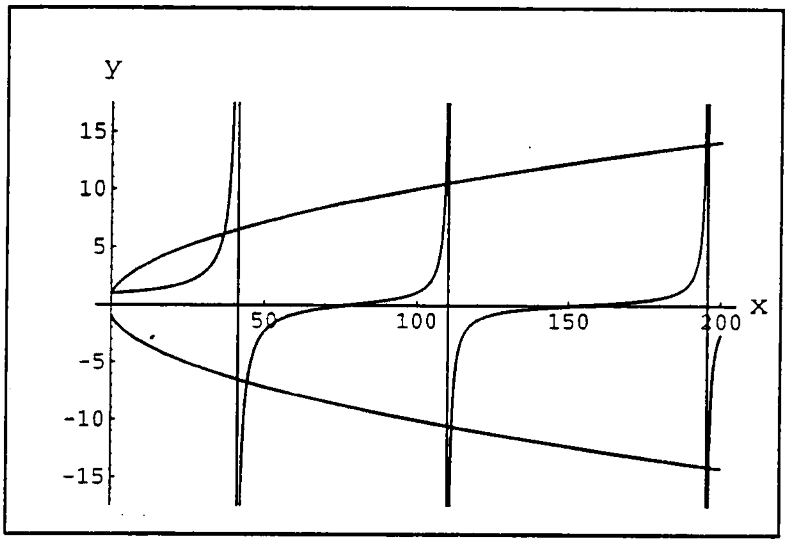


Figure 4.1: Determination of eigenfrequencies of inertia-gravity waves by two curves, $y = \phi(x)$ and $y^2 = 1 + \alpha x$, where $\alpha = gH/f^2a^2$. We choose $f = 2\Omega \sin 20°$, $H = 250$m and $a = 1000$km. (a) For wavenumber $s = 1$; and (b) for $s = 5$.

σ/f n \ s	-5	-4	-3	-2	-1	0	1	2	3	4	5
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1	6.758	5.682	4.595	3.490	2.348	1.000	1.508	2.737	3.899	5.029	6.138
2	10.625	9.392	8.130	6.827	5.459	3.953	5.389	6.732	8.023	9.279	10.509
3	14.052	12.749	11.415	10.040	8.608	7.087	8.581	9.999	11.365	12.694	13.994

Table 4.1: Analytically calculated eigenfrequencies of inertia-gravity waves for different wavenumbers and radial modes. Negative wavenumbers stand for the counterclockwise propagation inertia-gravity waves.

modes. There are two special cases: (1) when $n = 0$, the normalized frequency $\sigma/f = \pm 1$, which means the whole system oscillates with the inertial frequency $\pm f$; (2) when $s = 0$, the eigenfrequencies are still given by (4.18), but with κ determined from $J_0'(\kappa a) = 0$, the degenerate form of (4.16). The analytical dispersion relation is given as the solid line in Figure 4.2, keeping in mind that the geostrophic modes which have zero frequencies will all be overlapped the abscissa.

Substituting (4.17) into (4.9), we may solve for the other two eigenvectors. The results are

$$\begin{pmatrix} u' \\ v' \\ gh' \end{pmatrix} = \begin{pmatrix} [-\frac{fs}{r}J_s(\kappa r) + \kappa\sigma J_s'(\kappa r)]e^{i(s\phi-\sigma t-\pi/2)} \\ [\frac{\sigma s}{r}J_s(\kappa r) - \kappa f J_s'(\kappa r)]\, e^{i(s\phi-\sigma t)} \\ (\sigma^2 - f^2)J_s(\kappa r)e^{i(s\phi-\sigma t)} \end{pmatrix} \tag{4.20}$$

where the perurbation quantities have been normalized by a constant value of $gA/(\sigma^2 - f^2)$. From the formula we can see that the surface height h and the tangential wind v have the same phase, while the radial wind u is out of phase by a quarter of a cycle. Figures 4.3 and 4.4 show the eigenfunctions for some selected n and s.

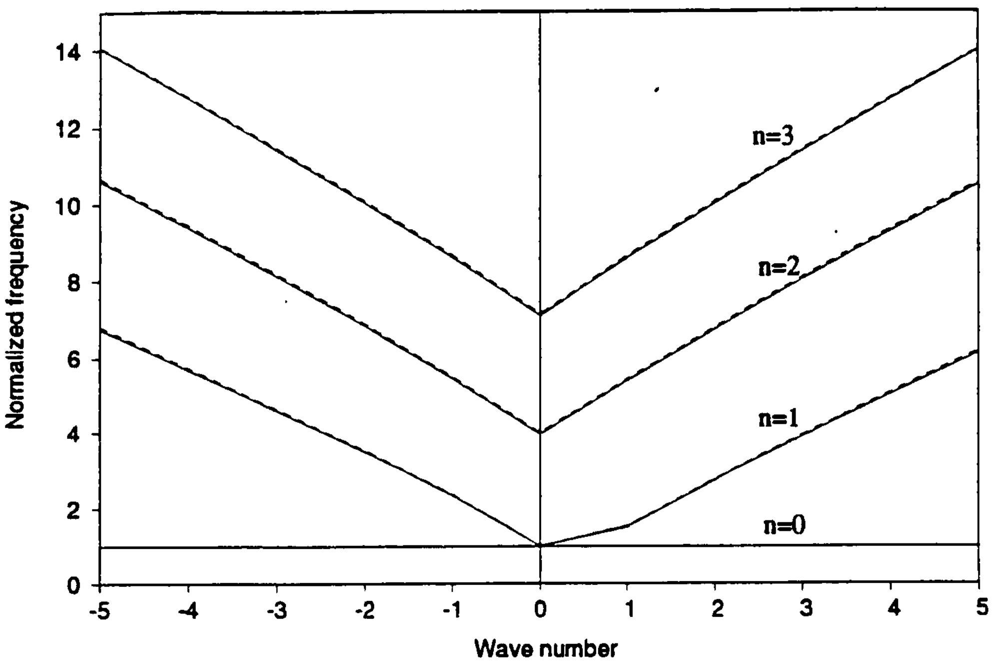


Figure 4.2: Dispersion diagram of inertia-gravity waves for the first four n modes and for wavenumbers 0 to 5. The eigenfrequencies have been normalized by f. The solid dispersion curves are analytically calculted, the dashed curves are numerically calculated.

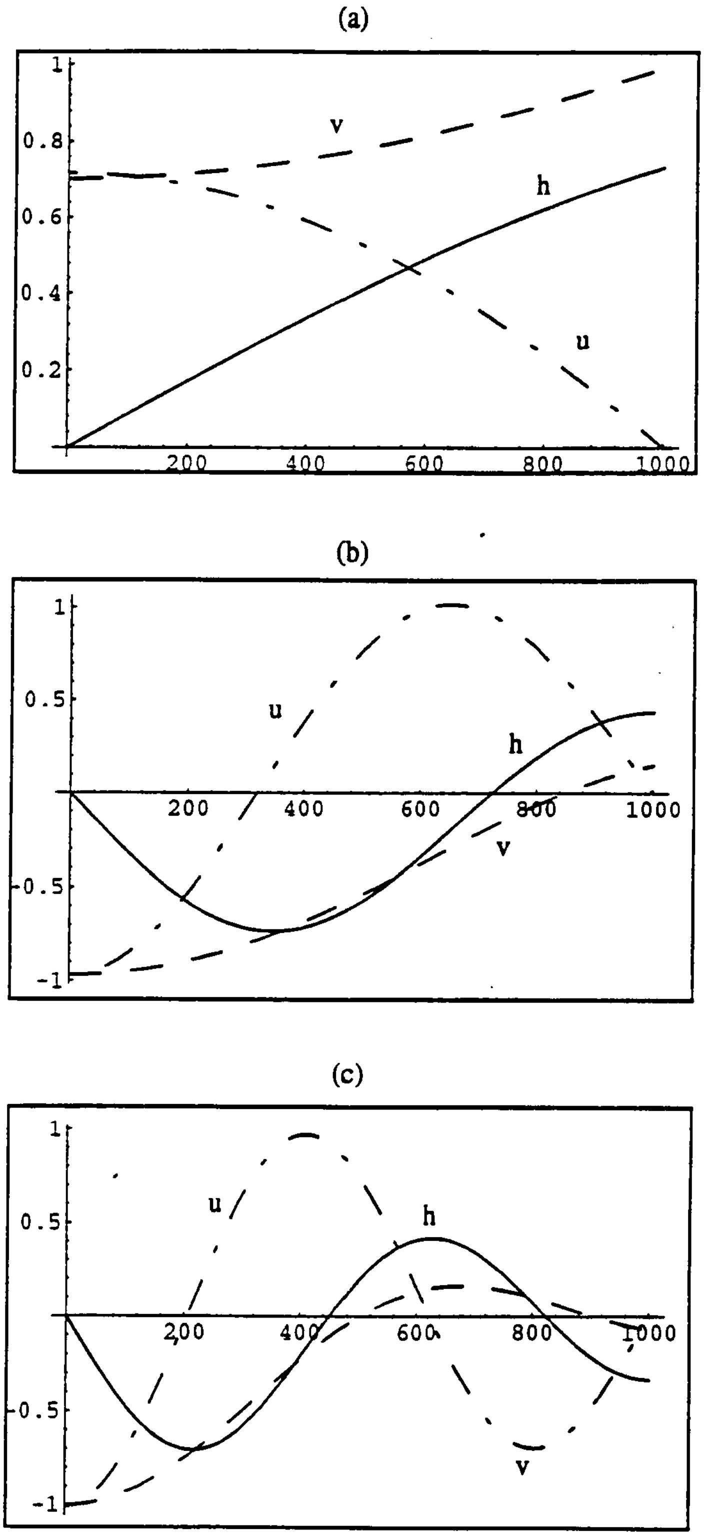
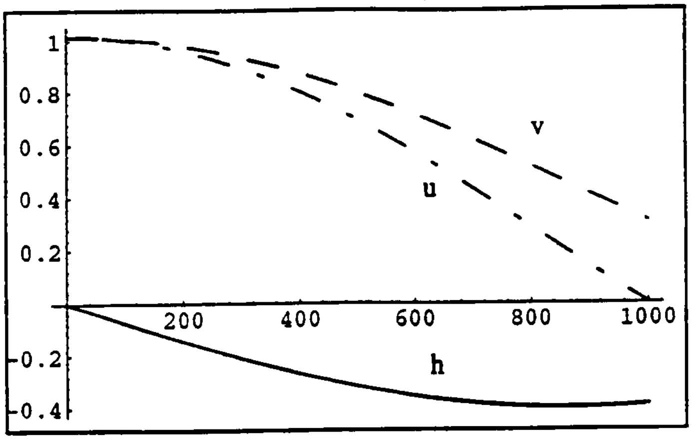
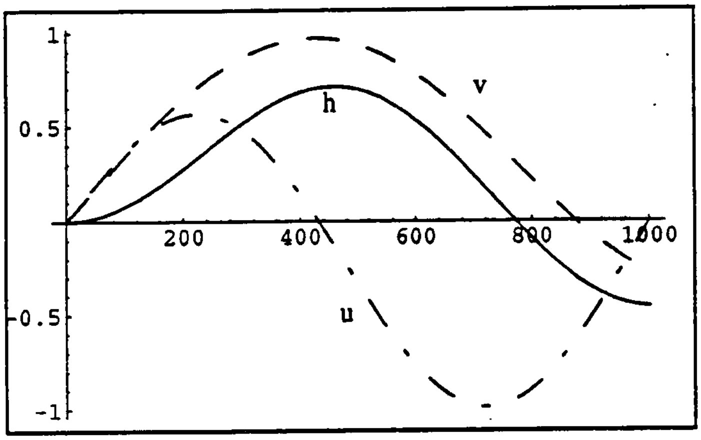


Figure 4.3: Analytical solution of winds and height surface corresponding to inertia-gravity waves as function of radius for different eigenmodes. They have all been nomalized by arbitrary values. (a) For n=1, s = 1; (b) for n = 2, s = 1; (c) for n = 3, s = 1.

(d)



(e)



(f)

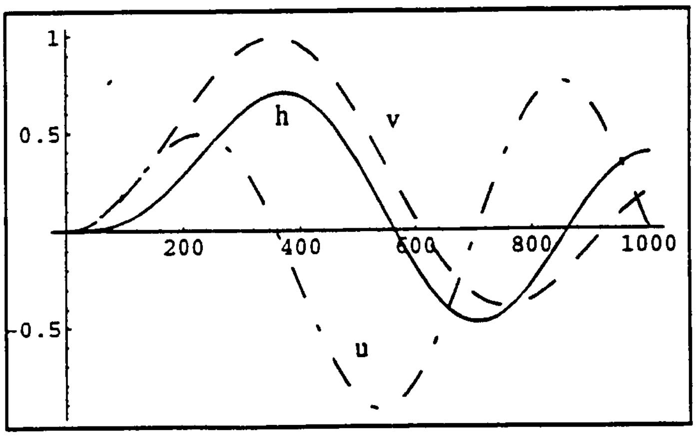


Figure 4.4: Same as Figure 4.3. (d) For $n = 1$, $s = -1$; (e) for $n = 2$, $s = 2$; (c) for $n = 3$, $s = 3$.

4.2 The vorticity (or PV) waves in Rankine's vortex

In passing, we have discussed the possible wave motions in a barotropic vortex with a resting basic state by using a set of linearized shallow-water primitive equations. In order to identify the nontrivial Rossby type of modes, a more complicated basic state has to be implemented. It is our experience, however, that even if the simplest nonresting basic state is adopted, the mathematical problem becomes so formidable that an analytical approach to the full spectrum analysis like the one in the previous section seems impossible. Before we go in quest of any numerical means, it is helpful for us to review some of the early analysis by Thomson (1880), and to develop some basic ideas for use in the later sections. The analytical results obtained from a simplified dynamical system, the nondivergent barotropic model, in studying the filtered spectra of wave motions in a barotropic vortex with the basic state of Rankine's type will provide some dynamical insights, in particular the concept of vorticity (or PV) waves.

4.2.1 Rankine's combined vortex

Of many basic circular flow profiles, Rankine's vortex is the simplest one, yet it is of meteorological relevance. It can be considered, on the first order of approximation, as a model of hurricanes. This will be illustrated by its velocity profile and pressure distributions. As shown in Figure 4.5, Rankine's vortex comprises a vortex core with a constant vorticity inside the core, and an irrotational outer region with zero vorticity. Mathematically, it can be represented by

$$\xi_n(r) = \begin{cases} \xi & 0 \le r \le a \\ 0 & a \le r \le \infty \end{cases}, \tag{4.21}$$

where ξ is a constant. Using Kelvin's circulation relation

$$\oint_l \mathbf{V} \cdot r d\phi = \iint_\sigma \mathbf{n} \cdot (\nabla \times \mathbf{V}) d\sigma = \iint_\sigma \xi_n d\sigma, \tag{4.22}$$

we may obtain by choosing two circuits, l_1 and l_2, inside and outside the vortex core

$$\oint_{l_1} \mathbf{V}_1 \cdot r_1 d\phi = \xi \pi r_1^2,$$

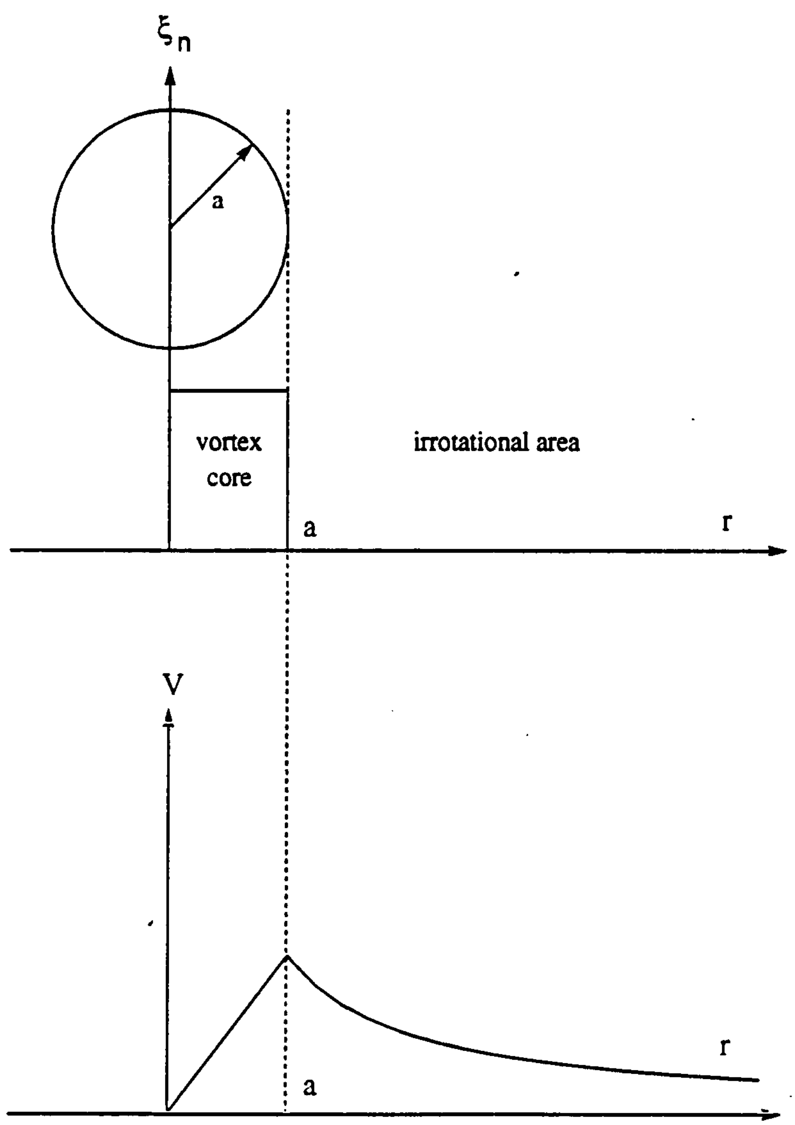


Figure 4.5: Schematic diagram of Rankine's combined vortex, and its vorticity and wind distributions.

$$\oint_{l_2} \mathbf{V_2} \cdot r_2 d\phi = \xi \pi a^2,$$

where $\mathbf{V} = V(r)\mathbf{e}_\phi$ is the axisymmetric tangential velocity, $r_1 < a < r_2$ the radii of the inner circuit, the vortex core and the outer circuit, respectively. Carrying out the circuit integrations, we may obtain the velocity profile in association with Rankine's combined vortex

$$V(r) = \begin{cases} \frac{1}{2}\xi r & 0 \le r \le a \\ \frac{1}{2}\xi \frac{a^2}{r} & a \le r \le \infty \end{cases}, \tag{4.23}$$

which, plotted in the lower panel of Figure 4.5, indicates that the tangential velocity is linearly increased as one moves from the vortex center outward, reaches its maximum value at the edge of vortex core, and then decreases at a $1/r$ rate. We can also find streamfunction of this axisymmetric flow by integrating $V = d\psi/dr$. It yields

$$\psi(r) = \begin{cases} \frac{1}{4}\xi(r^2 - a^2) & 0 \le r \le a \\ \frac{1}{2}\xi a^2 \ln\left(\frac{r}{a}\right) & a \le r \le \infty \end{cases}. \tag{4.24}$$

In order to investigate the mass distribution associated with Rankine's vortex, let us assume that the flow is steady. With this condition, it is straightforward to derive the shallow water version of Bernoulli equation from (4.1) and (4.2), which is in the form

$$\tfrac{1}{2}V^2 + gh = \text{const.} \tag{4.25}$$

On substituting the second entry of (4.23), we obtain the expression of surface height for outer core region

$$\tfrac{1}{8}\xi^2 \frac{a^4}{r^2} + gh = \text{const.},$$

where the constant can be determined by requiring $h \to H$, as $r \to \infty$. At this limit, we have

$$gh = gH - \tfrac{1}{8}\xi^2 \frac{a^4}{r^2}. \quad (r \ge a) \tag{4.26}$$

Inside the vortex core, we assume that the circular flow field is balanced by the gradient of surface height, giving the relation

$$g\frac{\partial h}{\partial r} = \left(f + \frac{V}{r}\right)V. \tag{4.27}$$

Integrating this equation, we have height field as a function of radius r within the core

$$gh = \tfrac{1}{4}\tilde{f}\xi r^2 + C$$

where $\tilde{f} = f + \frac{1}{2}\xi$, and C is a constant which must be chosen by matching the boundary condition at $r = a$. In doing so, this equation becomes

$$gh = gH - \tfrac{1}{4}\xi\eta a^2 \left[1 - \left(\frac{\tilde{f}}{\eta}\right)\frac{r^2}{a^2}\right]. \quad (r \leq a) \tag{4.28}$$

where $\eta = f + \xi$ is the absolute vorticity. In combining (4.26) and (4.28) together, we may write the distribution of geopotential in a Rankine's vortex as

$$gh = \begin{cases} gH - \frac{1}{4}\xi\eta a^2 \left[1 - \left(\frac{\tilde{f}}{\eta}\right)\frac{r^2}{a^2}\right] & , \quad 0 \leq r \leq a \\ gH - \frac{1}{8}\xi^2 \frac{a^4}{r^2} & \quad\;\; a \leq r \leq \infty \end{cases}, \tag{4.29}$$

where we can see that: (1) as $r \to \infty$, the surface height of the fluid is leveled out to a constant value H which is the approximate surface height of resting basic state; (2) at the vortex center $r = 0$, the geopotential reaches its lowest value gh_0, where $gh_0 = \frac{1}{4}\xi\eta a^2$. The faster Rankine's vortex spins, the larger gh_0 becomes, which represents a stronger low center system. A transient development of tropical cyclones can be depicted by the continuous strengthening of such a Rankine's vortex. Three typical stages along this development have been classified in Elsberry *et al.* (1985) as tropical depressions, tropical storms and typhoons or hurricanes. They are all characterized by definite closed isobars or height surfaces with tangential circulation around the low center.

	r	V	ξ	R_{oc}
Tropical depressions	200 km	10 ms^{-1}	$0.5 \times 10^{-4}s^{-1}$	1
Tropical storms	100 km	30 ms^{-1}	$3 \times 10^{-4}s^{-1}$	6
Typhoons/Hurricanes	33.3 km	45 ms^{-1}	$13.5 \times 10^{-4}s^{-1}$	27

Table 4.2: Classification of different development stages of tropical cyclones. From the definition by Elsberry *et al.* (1985).

In Table 4.2, we list some characteristic values of radii of the disturbances, tangential winds, relative vorticities and the curvature Rossby numbers for such three development stages.

With the values in Table 4.2, we can plot the pressure or surface height distributions, shown in the upper panel of Figure 4.6, for tropical depressions, tropical storms and hurricanes by using (4.29). A three-dimensional view of tropical cyclone system is also plotted in the lower panel of Figure 4.6.

The above analysis can naturally be extended to a stratified fluid system. In that case, the tangential flow will have the exactly same profile as that in shallow water system, while the surface height distribution in current analysis will be replaced by the pressure distribution which much resemble the low-core system of hurricanes.

4.2.2 The nondivergent barotropic wave dynamics

Let us begin with the set of nondivergent shallow water equations in the form

$$\frac{\partial u}{\partial t} - \zeta v + \frac{\partial}{\partial r}[gh + \tfrac{1}{2}(u^2 + v^2)] = 0, \tag{4.30}$$

$$\frac{\partial v}{\partial t} + \zeta u + \frac{\partial}{r\partial \phi}[gh + \tfrac{1}{2}(u^2 + v^2)] = 0, \tag{4.31}$$

$$\frac{\partial (ru)}{r\partial r} + \frac{\partial v}{r\partial \phi} = 0, \tag{4.32}$$

where ζ is the absolute vorticity. Since u and v are diagnostically related through the nondivergent condition (4.31), the first two equations can no longer predict the wind fields independently. Therefore, (4.30) and (4.31) must be reformulated under the constraint (4.32), which results in a single predictive equation, the vorticity equation, in the form

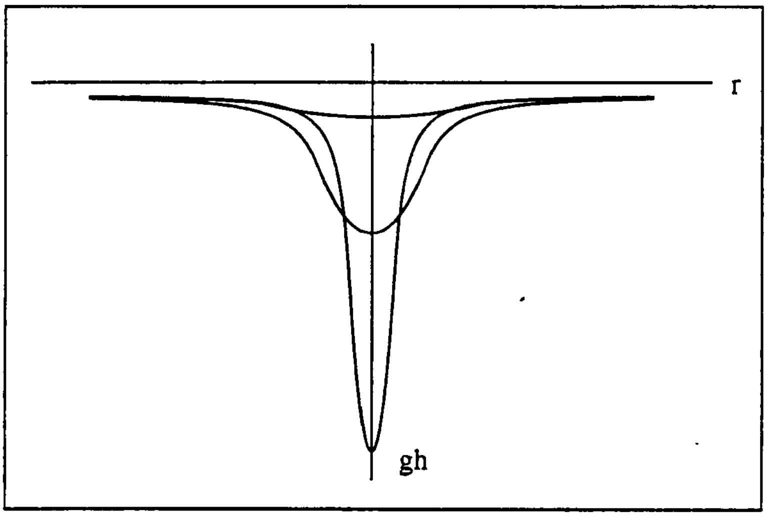
$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial r} + v\frac{\partial \zeta}{r\partial \phi} = 0, \tag{4.33}$$

where u, v and ζ are all diagnosed from the single advective variable ψ through the following inversion operators

$$u = -\frac{\partial \psi}{r\partial \phi}, \quad v = \frac{\partial \psi}{\partial r}, \tag{4.34}$$

$$\zeta = f + \nabla^2\psi = f + \frac{\partial}{r\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{\partial^2 \psi}{r^2\partial \phi^2}. \tag{4.35}$$

(a)



(b)

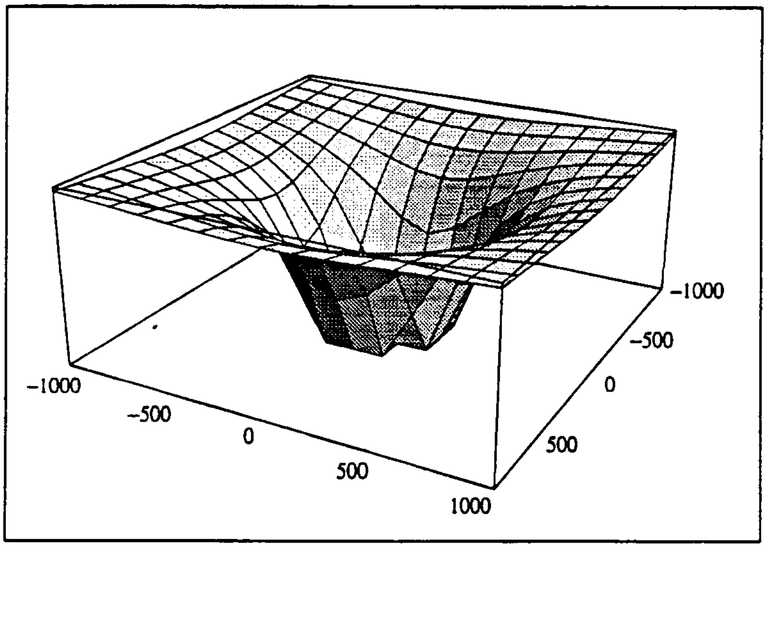


Figure 4.6: Surface height distribution in Rankine's vortex calculated from (4.29). (a) The distributions for tropical cyclone at three development stages. (b) A three-dimensional view of height distribution for a tropical storm case.

Note that the closed system of equations (4.33), (4.34) and (4.35) forms a balanced dynamical model, the so-called "nondivergent barotropic model". There is only one class of eigenfrequencies retained in this system as a result of one independent predictive equation. Rossby (1939) used this model with the β-plane approximation to promulgate the existance of a type of low frequency waves. These waves later were named after him, and are known as "Rossby waves". The restoring mechanism for this type of waves is the β-effect, i.e., the northward gradient of vorticity in the basic flow. The generalization of this type of wave dynamics to full spherical geometry was straightforwardly done by Haurwitz(1940).

Along the same line on an f-plane, however, if one linearizes this system about a basic state of rest and solves the eigenvalue problem, the only eigenfrequency that he or she would get is the stationary geostrophic mode, which is consistent with the full spectrum analysis using the primitive dynamical system discussed in the previous section.

We now employ a more complicated, nonresting basic state to investigate this problem further. In particular, let us linearize (4.33)–(4.35) about a basic state of Rankine's vortex. The resultant linearized vorticity equation is

$$\frac{\partial}{\partial t}\nabla^2\psi' + \bar{V}\frac{\partial}{r\partial\phi}\nabla^2\psi' = 0, \tag{4.36}$$

where $\bar{V}$ is given by (4.23). Note that this equation holds everywhere in the model domain except at $r = a$, where the discontinuity in the vorticity field occurs. We next substitute a wave perturbation

$$\psi'(r,\phi,t) = \Psi(r)e^{i(s\phi-\sigma t)} \tag{4.37}$$

into (4.36) to obtain

$$\left(\frac{\bar{V}}{r}s - \sigma\right)\left[\frac{d}{rdr}\left(r\frac{d\Psi}{dr}\right) - \frac{s^2}{r^2}\Psi\right] = 0.$$

In searching for solutions of nonsolid-body rotation, $\sigma \neq s\bar{V}/r$, (4.37) must yield

$$r\frac{d}{dr}\left(r\frac{d\Psi}{dr}\right) - s^2\Psi = 0, \tag{4.38}$$

which has general solution of the form

$$\Psi(r) = \mathrm{A}r^s + \frac{\mathrm{B}}{r^s}. \tag{4.39}$$

Due to the finiteness of the solution when $r \to 0$, and $r \to \infty$, (4.39) is reduced to its proper form

$$\Psi(r) = \begin{cases} \mathrm{A}r^s & 0 \le r < a \\ \mathrm{B}/r^s & a < r < \infty \end{cases}, \tag{4.40}$$

where A and B are two constants which must be determined by boundary conditions at $r = a$. The first boundary condition that we consider here is that the radial velocity $u = -\partial\psi/r\partial\phi$ must be continuous, which gives the condition that $\mathrm{A}a^s = \mathrm{B}/a^s$, so that (4.40) becomes

$$\Psi(r) = \mathrm{D}\begin{cases} (r/a)^s & 0 \le r \le a \\ (a/r)^s & a \le r < \infty \end{cases}, \tag{4.41}$$

where $\mathrm{D} = \mathrm{A}a^s = \mathrm{B}/a^s$ is another constant, which may be determined by the continuity of tangential velocity $v = \partial\psi/\partial r$ at $r = a$

$$\tfrac{1}{2}\xi r^2 + sD\left(\frac{r}{a}\right)^s e^{i(s\phi-\sigma t)} = \tfrac{1}{2}\xi a^2 - sD\left(\frac{a}{r}\right)^s e^{i(s\phi-\sigma t)}$$

where ξ, defined in (4.21), is the constant vorticity of Rankine's vortex core. We now evaluate this equation at $r = a + \eta$, where $\eta(\phi,t) = \hat{\eta}e^{i(s\phi-\sigma t)}$ is the particle radial displacement, and linearize it. In so doing, the above equation gives

$$sD = -\tfrac{1}{2}\xi a\hat{\eta}. \tag{4.42}$$

Next we need to consider a dynamical boundary condition in order to determine the eigenvalue appeared in the general solution. This condition may be given by considering the continuity of fluid which indicates that every surface inside the fluid, including the interface at $r = a$, is a material surface, so that the radial velocity associated with particle displacement at $r = a$ must be equal to the radial movement of the interface itself, i.e.,

$$\left[\frac{\partial\eta}{\partial t} + \bar{V}\frac{\partial\eta}{r\partial\phi}\right]_{r=a} = [u']_{r=a}. \tag{4.43}$$

When substituted into the expressions for η and u', (4.43) yields

$$-\sigma\hat{\eta} + \frac{\bar{V}}{a}s\hat{\eta} = -\frac{s\mathrm{D}}{a} = \tfrac{1}{2}\zeta\hat{\eta}.$$

After rearranging terms, we finally get the dispersion relation

$$\sigma = \tfrac{1}{2}\xi(s-1). \tag{4.44}$$

This relation obviously describes wave motions of Rossby type, because (4.44) can easily be rewritten into the well-known form: $\sigma = \bar{\omega}s - \beta s/s^2$, the Rossby wave disperssion relation, where now the β term is $\frac{1}{2}\xi s$. As the restoring mechanism, the β effect here may be considered to bear the same physical interpretation as in Rossby's case in the sense that $\frac{1}{2}\xi$ approximates the radial gradient of vorticity at basic state across the interface at $r = a$, although this gradient is infinitely large in this particular mathematical problem. If we tried to think about this problem in a more physical situation, where the vorticity jump from a finite value to zero must occur over a finite spatial distance, $\frac{1}{2}\xi$ would represent some averaged gradient over a unit distance. Another interpretation is that $\frac{1}{2}\xi$ is related to the nonresting basic tangential flow, and therefore the radial gradient of surface height (or more generally, the pressure gradient) is induced and maintained, which may certainly act as the restoring force. In any rate, waves can be generated or propagated in: (1) any flow field endowed with the gradient in its vorticity field, or in its potential vorticity (PV) field for more general case of compressibe, stratified fluid; (2) any nonresting flow field which generates a gradient of mass distribution. It is understandable that such waves constitute a more general class of wave motions than Rossby waves. In other words, the Rossby waves arised from the vorticity (PV) gradient due to the earth's geometry (or the slope of bottom topography, in the context of oceanography) are just deligated a subclass of these generalized waves. Here we call these waves the vorticity waves, or PV waves.

It is of no surprise, by the same token, that there exists only one single radial mode (or n mode) of these waves in this nondivergent barotropic model. The vorticity wave can only be initiated and propagated along the discontinuous interface at $r = a$, now that the vorticity gradient is zero almost everywhere and is singular at one site. That the dynamical system under nondivergent conditions perceives and maintains this vorticity discontinuity results in the "solitarization" of vorticity waves. This is, however, not the case in the models with divergent flows, where the PV gradient or the gradient of potential pseudo-height may vary continuously, regardless of the jump in the vorticity field. Therefore, the PV waves may present a family of n modes (e.g., the primitive equation model in the next section and the mixed balanced model in the next chapter).

Note also that (4.44) gives an oscillation with a much higher frequency than that of Rossby waves in a common sense. Table 4.3 lists the normalized frequencies calculated from (4.44) in terms of different developing stages of tropical cyclones. The frequency of this generalized Rossby wave is proportional to the rotational rate of a circular vortex. As the low-centered system deepens, and the vortex spins up, the low-frequency Rossby waves can be translated to the wave motions with high frequencies. For example, the frequencies shown in Table 4.3 for a mature hurricane can be as high as a hundred times the Coriolis parameter, and as shown in the next section, such frequencies are even higher than the highly oscillatory inertia-gravity modes.

		$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
	Tropical depressions	0	1.00	2.00	3.00	4.00
σ/f	Tropical storms	0	6.02	12.04	18.06	24.08
	Typhoons/Hurricanes	0	27.09	54.18	81.28	108.37

Table 4.3: Normalized frequencies of generalized Rossby waves predicted by the nondivergent barotropic model.

Under this analysis, generally speaking of Rossby wave motions representing the slow motions in the atmosphere and ocean may not be a quite scientifically founded statement. By the same token, generally constructing a model to filter all the fast modes may not be a proper procedure for studying certain types of problems.

The dispersion relation (4.44) is plotted in Figure 4.7, with the values listed in Table 4.3. All the dispersion curves are shifted along the wavenumber axes by one unit, with wavenumber 1 being the stationary mode, and wavenumber 0 corresponding to a negative frequency whose physical interpretation is not clear. On the other hand, the wavenumber 0 motion (i.e., axisymmetric expansion or contraction of the vortex) in an incompressible fluid system is not allowed. This can be seen from (4.32), where for zero-wavenumber motion, $ru = const.$ for all r. But the constant is zero (determined by the condition at the vortex center). Thus, in order for all r that $ru = 0$ holds, u must be zero. This may partially explain why the dispersion curves are shifted along the wavenumber axis. The

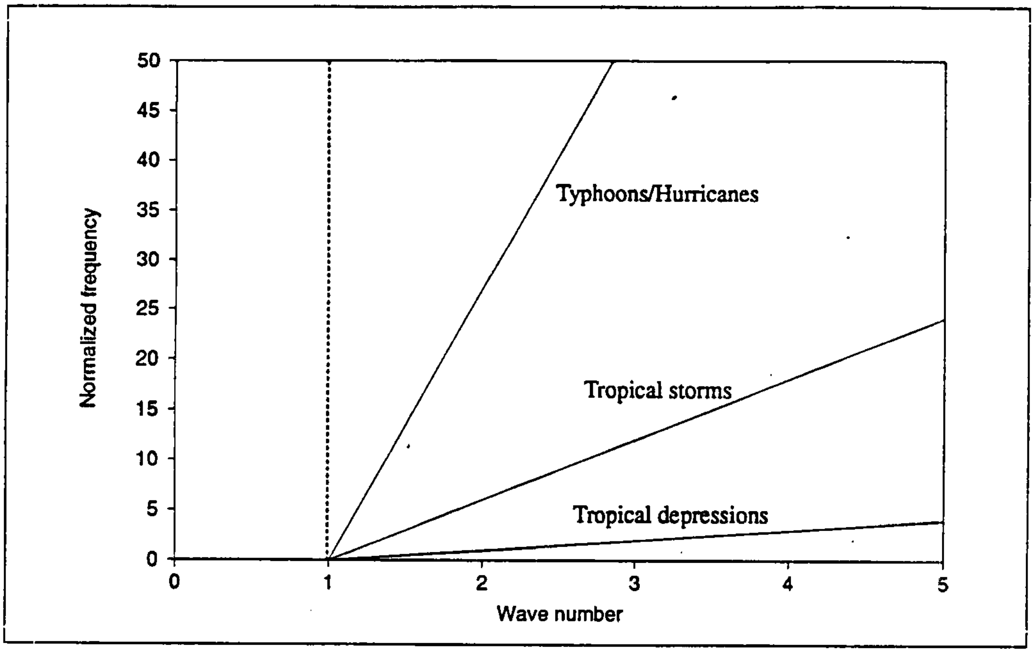


Figure 4.7: Dispersion diagram of generalized Rossby waves for wavenumber 1 to 5. Different curve is for different intensity of Rankine vortex represented different development stage of a tropical cyclone.

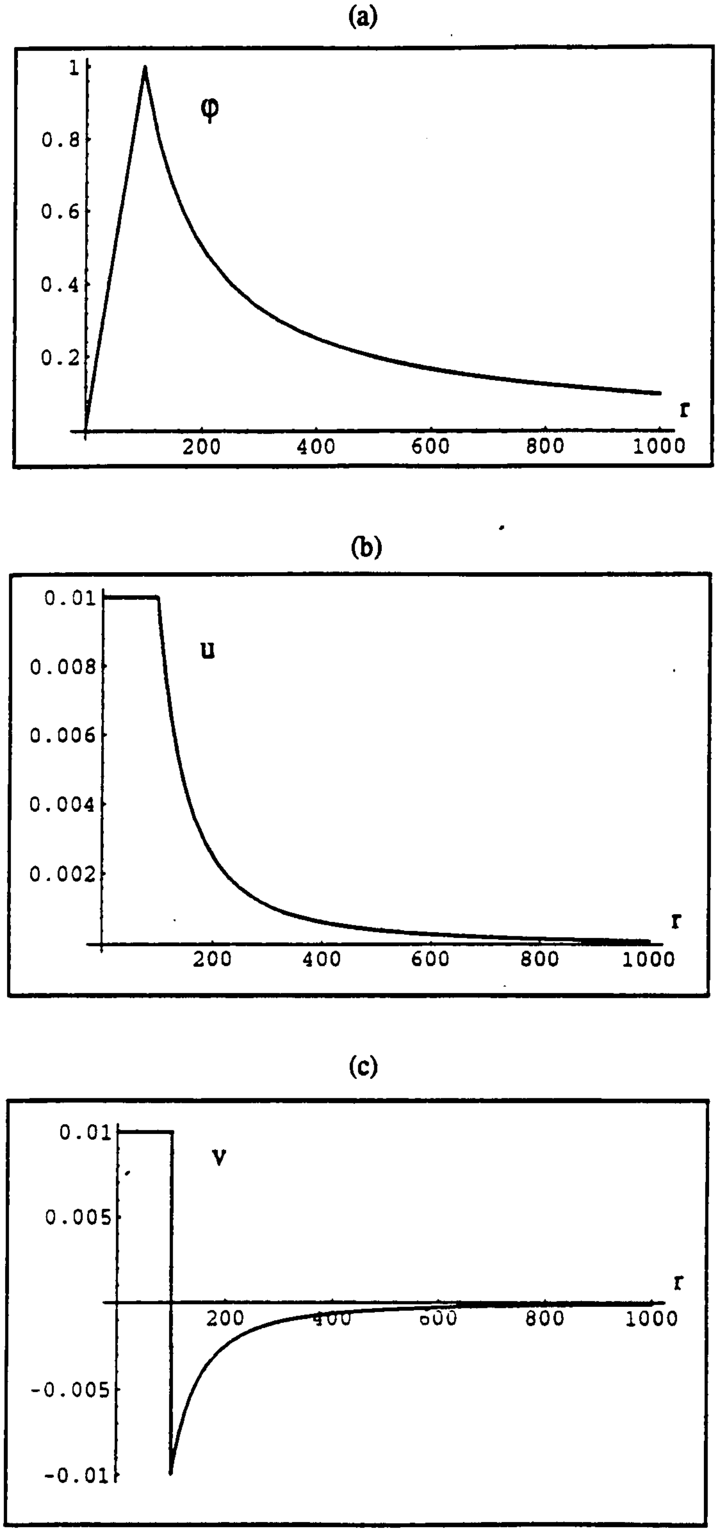


Figure 4.8: Analytical solution of winds and streamfunction of generalized Rossby waves as functions or radius, calculated from a nondivergent barotropic model. The case shown here is for a = 100km and $s = 1$.

steepest line correspons to the strongest vortex. The eigenfunction is given by (4.41) and u and v fields are retrived through (4.34) as

$$u' = \frac{s}{r}\begin{cases} (r/a)^s\, e^{i(s\phi-\sigma t-\pi/2)} \\ (a/r)^s\, e^{i(s\phi-\sigma t-\pi/2)} \end{cases}; \tag{4.45}$$

and

$$v' = \frac{s}{r}\begin{cases} (r/a)^s\, e^{i(s\phi-\sigma t)} \\ -(a/r)^s\, e^{i(s\phi-\sigma t)} \end{cases}, \tag{4.46}$$

where u' and v' have been normalized by the constant D. Again we see that the stream-function and tangential velocity have the same phase, while the radial velocity has phase sifting by one quarter of a cycle. The eigenfunctions corresponded with the tropical storm case for $s = 1$ are plotted in Figure 4.8. We note that the discontinuity occurs in the tangential velocity field.

4.3 Free oscillations in Rankine's vortex

In this section, we will use a linearized primitive equation model to conduct a full spectral analysis of unforced wave motions superimposed on an axisymmetric flow field of Rankine's type. The model is a modified version of that of Stevens and Ciesielski (1984), and was used by Flatau and Stevens (1989) to study the barotropic and inertial instability in the hurricane outflow layer. A brief discription of this model is as follows.

Let us consider a set of shallow water primitive equations of the form (4.1)-(4.4). We now linearize this set about a basic state of axisymmetry, i.e., $(\bar{u},\bar{v},\bar{h}) = (0,\bar{v}(r),\bar{h}(r))$. In so doing, we obtain

$$\left(\frac{\partial}{\partial t}+\bar{v}\frac{\partial}{r\partial\phi}\right)u' - \left(f+\frac{2\bar{v}}{r}\right)v' + g\frac{\partial h'}{\partial r} = 0, \tag{4.47}$$

$$\left(\frac{\partial}{\partial t}+\bar{v}\frac{\partial}{r\partial\phi}\right)v' + \left(f+\frac{\partial(r\bar{v})}{r\partial r}\right)u' + g\frac{\partial h'}{r\partial\phi} = 0, \tag{4.48}$$

$$\left(\frac{\partial}{\partial t}+\bar{v}\frac{\partial}{r\partial\phi}\right)h' + \bar{h}\left(\frac{\partial(ru')}{r\partial r}+\frac{\partial v'}{r\partial\phi}\right) + u'\frac{\partial\bar{h}}{\partial r} = 0. \tag{4.49}$$

When we substitute a wave perturbation of the form $x' = \hat{x}(r)\exp i(s\phi-\sigma t)$, where x denotes u, v or h, into this system, (4.47)-(4.49) becomes

$$
\begin{pmatrix}
-\hat{\sigma} & \tilde{f} & -g\frac{d}{dr} \\
\bar{\eta} & -\hat{\sigma} & g\frac{s}{r} \\
\frac{\bar{h}}{r}\frac{dr()}{dr} + \frac{d\bar{h}}{dr} & s\frac{\bar{h}}{r} & -\hat{\sigma}
\end{pmatrix}
\begin{pmatrix} -i\hat{u} \\ \hat{v} \\ \hat{h} \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.50}
$$

where we have defined the Doppler-shifted frequency $\hat{\sigma}$, absolute vorticity of the mean flow $\bar{\eta}$, and modified coriolis parameter $\tilde{f}$ as:

$$
\hat{\sigma} \equiv \sigma - \frac{s\bar{v}}{r}, \tag{4.51}
$$

$$
\bar{\eta} \equiv f + \frac{\partial(r\bar{v})}{r\partial r}, \tag{4.52}
$$

$$
\tilde{f} \equiv f + \frac{2\bar{v}}{r}, \tag{4.53}
$$

respectively. Noting that the Doppler-shifted frequency $\hat{\sigma}$ is a function of r, we rewrite (4.50) in the following form:

$$
\begin{pmatrix}
s\frac{\bar{v}}{r} & \tilde{f} & -g\frac{d}{dr} \\
\bar{\eta} & s\frac{\bar{v}}{r} & g\frac{s}{r} \\
\frac{\bar{h}}{r}\frac{dr()}{dr} + \frac{d\bar{h}}{dr} & s\frac{\bar{h}}{r} & s\frac{\bar{v}}{r}
\end{pmatrix}
\begin{pmatrix} -i\hat{u} \\ \hat{v} \\ \hat{h} \end{pmatrix}
= \sigma
\begin{pmatrix} -i\hat{u} \\ \hat{v} \\ \hat{h} \end{pmatrix}. \tag{4.54}
$$

This gives an eigenvalue problem

$$
\mathbf{AX} = \sigma \mathbf{X}, \tag{4.55}
$$

where σ denotes the eigenvalues, and $\mathbf{X}$ the corresponded eigenvectors. Equation (4.54) is first discretized in the radial direction on a staggered grid with $\hat{h}$ defined at points $r = n\Delta r$, and $\hat{u}$ and $\hat{v}$ defined at $r = (n + 1/2)\Delta r$ points where $n = 0, 1, 2...$, and Δr is the grid spacing. We then solve the discretized system as an eigenvalue problem by using the EISPACK routines (Smith *et al.*, 1976).

4.3.1 Numerical calculations of eigenfrequencies and eigenfunctions in barotropic vortices with a resting basic state

In order to test the credibility of the numerical model, we first run the model with a resting basic state, then compare the model output with the analytical results discussed in section 4.1.

The boundary conditions which are compatible with the analytical case are:

σ/f (n \ s)	-5	-4	-3	-2	-1	0	1	2	3	4	5
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1	6.812	5.729	4.632	3.518	2.365	1.000	1.525	2.765	3.938	5.078	6.196
2	10.681	9.449	8.186	6.878	5.503	3.993	5.433	6.784	8.080	9.338	10.568
3	14.071	12.782	11.459	10.091	8.661	7.138	8.634	10.050	11.410	12.729	14.015

Table 4.4: Numerically calculated eigenfrequencies of inertia-gravity waves for different wavenumbers and radial modes. Negative wavenumbers stand for counterclockwise propagation inertia-gravity waves.

1. At the outer boundary, we assume no perturbation radial wind, i.e.,

   $u' = 0;$

2. At the vortex center ($r = 0$), the finite condition is required in perturbation surface height field, which gives:

   $h' = 0$ for $s \neq 0$,

   $dh'/dr = 0$ for $s = 0$.

In Table 4.4, we listed the calculated eigenfrequencies for the first four n modes. They are identified as the frequencies of waves of the first class, the inertia-gravity waves, after compared with those in Table 4.1. The results are so consistant with the analytical solutions that one can hardly distinquish the differences in their dispersion relation diagram (ref. Figure 4.2: the dashed lines are for the numerical calculations). The geostrophic modes (not listed in this table) all have zero frequencies as expected. We also plot selected eigenfunctions in Figure 4.9.

4.3.2 Numerical calculations of eigenfrequencies and eigenfunctions with a basic state of Rankine's vortex

We now specify the basic state of axisymmetry to have the radial structure of Rankine's vortex, as discussed in the previous section. The intensity of the vortex is chosen so as to represent the three different stages of a tropical cyclone's evolution. The inner

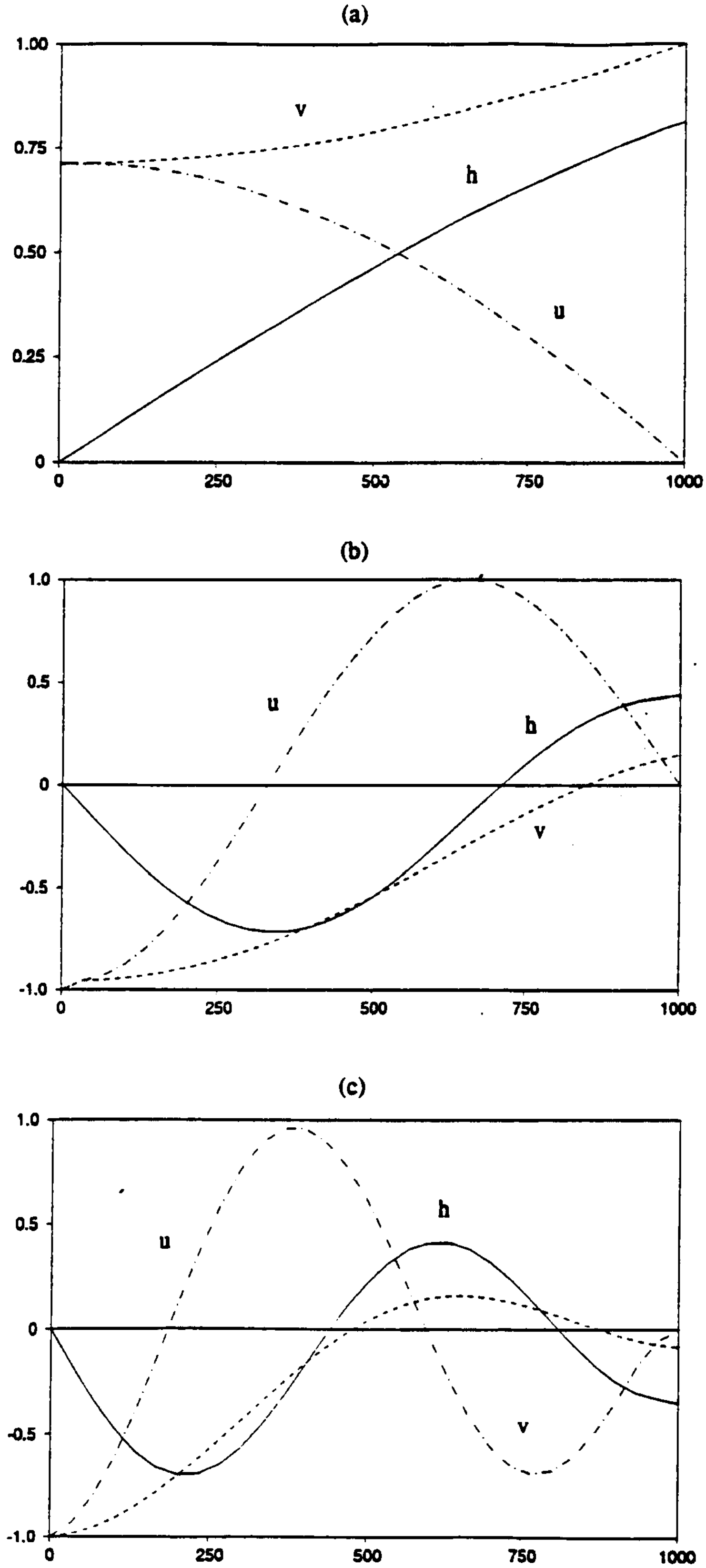


Figure 4.9: Numerical solution of winds and height surface corresponding to inertia-gravity waves as function of radius for different eigenmodes. They have all been nomalized by arbitrary values. (a) For n=1, s = 1; (b) for n = 2, s = 1; (c) for n = 3, s = 1.

boundary condition is the same as that of the "resting-basic-state" run, whereas the outer boundary condition is modified to assume vanishing of perturbation surface height instead, i.e., $h' = 0$ at $r = 1000$km.

The model results are first summarized in Tables 4.5–4.7. The eigenfrequencies are listed in terms of the inertia-gravity waves and the generalized Rossby waves for different wavenumbers and radial modes. The dispersion relation diagram is shown in Figure 4.10 for the first three radial modes both for inertia-gravity and generalized Rossby waves. The panel from top to bottom is each for tropical depression, tropical storm and hurricane cases. We can see from these diagrams that in the early developing stage of a tropical cyclone (e.g., a tropical depression) when the rotation of the vortex is slower, and the vortex is larger, Rossby waves are primarily presented as the slow motions, much slower than the inertial-gravity oscillations.

	Inertia-gravity modes					Rossby modes				
	$n = 1$	2	3	4	5	n=1	2	3	4	5
s =-5	8.78	12.17	15.31	18.34	21.29	–	–	–	–	–
-4	7.60	10.91	14.02	17.06	20.06	–	–	–	–	–
-3	6.40	9.63	12.74	15.82	18.87	–	–	–	–	–
-2	5.18	8.37	11.51	14.62	17.69	–	–	–	–	–
-1	4.08	7.20	10.30	13.41	16.52	–	–	–	–	–
0	3.01	5.87	9.06	12.23	15.36	0.00	0.00	0.00	0.00	0.00
1	4.24	7.56	10.84	14.07	17.24	1.00	0.72	0.56	0.44	0.36
2	5.68	9.17	12.52	15.80	19.00	2.00	1.44	1.10	0.98	0.89
3	7.03	10.65	14.10	17.43	20.67	3.00	2.09	1.69	1.33	1.08
4	8.34	12.07	15.58	18.96	22.24	4.01	2.91	2.25	1.77	1.44
5	9.62	13.44	17.00	20.42	23.72	5.01	3.76	2.81	2.22	1.80

Table 4.5: Numerically calculated eigenfrequencies for a basic vortex of the tropical depression case.

As the vortex spins up and shrinks, the frequencies of these Rossby waves become higher and higher, and they eventually exceed those of the inertia-gravity waves. This result is qualitatively consistent with the analytical solution obtained from the nondivergent barotropic model, where the dispersion relation (4.44) states that the frequency of Rossby waves is proportional to the rotation rate of the vortex. On the other hand, as

	Inertia-gravity modes					Rossby modes				
	$n = 1$	2	3	4	5	n=1	2	3	4	5
s =-5	10.10	13.96	17.50	20.83	23.96	–	–	–	–	–
-4	8.72	12.47	15.93	19.18	22.23	–	–	–	–	–
-3	7.32	10.94	14.31	17.53	20.70	–	–	–	–	–
-2	5.88	9.40	12.82	16.21	19.58	–	–	–	–	–
-1	4.73	8.37	11.93	15.39	18.79	–	–	–	–	–
0	3.16	6.84	10.57	13.03	17.96	0.00	0.00	0.00	0.00	0.00
1	4.96	8.89	12.77	16.59	19.04	5.98	3.14	2.07	1.46	1.08
2	6.69	10.83	14.83	18.74	22.55	11.97	5.88	4.31	2.96	2.18
3	8.29	12.58	16.67	20.64	24.51	17.98	10.39	6.41	4.46	3.28
4	9.84	14.25	18.40	22.41	26.30	23.99	14.86	8.58	5.96	4.38
5	11.35	15.88	20.09	24.12	28.03	30.01	17.50	10.76	7.47	5.49

Table 4.6: Numerically calculated eigenfrequencies for a basic vortex of the tropical storm case.

	Inertia-gravity modes					Rossby modes				
	$n = 1$	2	3	4	5	n=1	2	3	4	5
s =-5	11.96	16.67	21.06	25.26	29.33	–	–	–	–	–
-4	10.35	14.95	19.28	23.45	27.51	–	–	–	–	–
-3	8.71	13.19	17.45	21.59	25.62	–	–	–	–	–
-2	7.02	11.35	15.54	19.63	23.64	–	–	–	–	–
-1	5.37	9.68	13.96	18.20	22.39	–	–	–	–	–
0	3.53	7.78	12.09	16.39	20.64	0.00	0.00	0.00	0.00	0.00
1	5.53	9.97	14.38	18.74	23.04	–	–	2.84	1.65	1.06
2	7.42	12.06	16.54	20.96	25.30	–	15.46	5.74	3.29	2.12
3	9.19	13.97	18.54	23.01	27.36	–	22.69	8.82	4.99	3.20
4	10.90	15.82	20.45	24.93	29.29	108.78	29.91	11.99	6.71	4.29
5	12.58	17.61	22.30	26.81	31.18	126.21	38.02	15.24	8.45	5.39

Table 4.7: Numerically calculated eigenfrequencies for a basic vortex of the hurricane case.

the vortex rotates faster, the inertial stability factor increases, and thus the inertial oscillation is enhanced. This process tends to increase the frequency of inertia-gravity waves considerably. However, the restoring force for gravitational oscillations is related to gh. As the vortex rotates faster, the vortex becomes shallower near the center. This process tends to exert a counter-effect on the increase of the frequency of inertia-gravity waves. The combined effect of both processes may not make as substantial a change in frequency for inertia-gravity waves as for Rossby waves. This seems to be the case shown in the dispersion diagram. The high-frequency Rossby waves, unlike high-frequency inertia-gravity waves, can be dynamically important in atmospheric and oceanic motions, because they represent the components of the highly rotational motion itself.

In Figure 4.11, we plot some selected eigenfunctions of inertia-gravity waves for the tropical storm case. The three panels in this figure show the eigenvectors of the first, second and third radial modes for azimuthal wavenumber 1. The unbalanced flows and mass is distributed over the whole domain with the strongest influence of basic flow in the vicinity of the edge of Rankine's vortex core. All the fields are still continuous functions of r, but the nonsmooth feature is seen in the tangential velocity field.

The eigenfunction diagrams for Rossby waves are shown in Figures 4.12 and 13. Both the balanced mass and flow fields are concentrated within or near the Rankine vortex. As we go from low radial modes to the higher ones (from the top down: n=1,2,...,6), the concentrated blob of u, v and h fields begin to spread out, and the discontinuous interface of vorticity moves outward from the edge of Rankine's vortex. The enveloped shapes of these eigenvectors are very similar to those derived from the nondivergent barotropic model. An exceptional feature is presented in $s = 1$, $n = 1$ mode, where height surface does not go to zero as $r \to 0$. This mode may represent a computational mode.

Another problem associated with this model simulation is that the resolution in the radial direction may not be good enough, especially inside the Rankine vortex core. This problem may result in missing some physically important eigenmodes, e.g., those unresoled modes listed with dashes in the Rossby frequency manifold in Table 4.7.

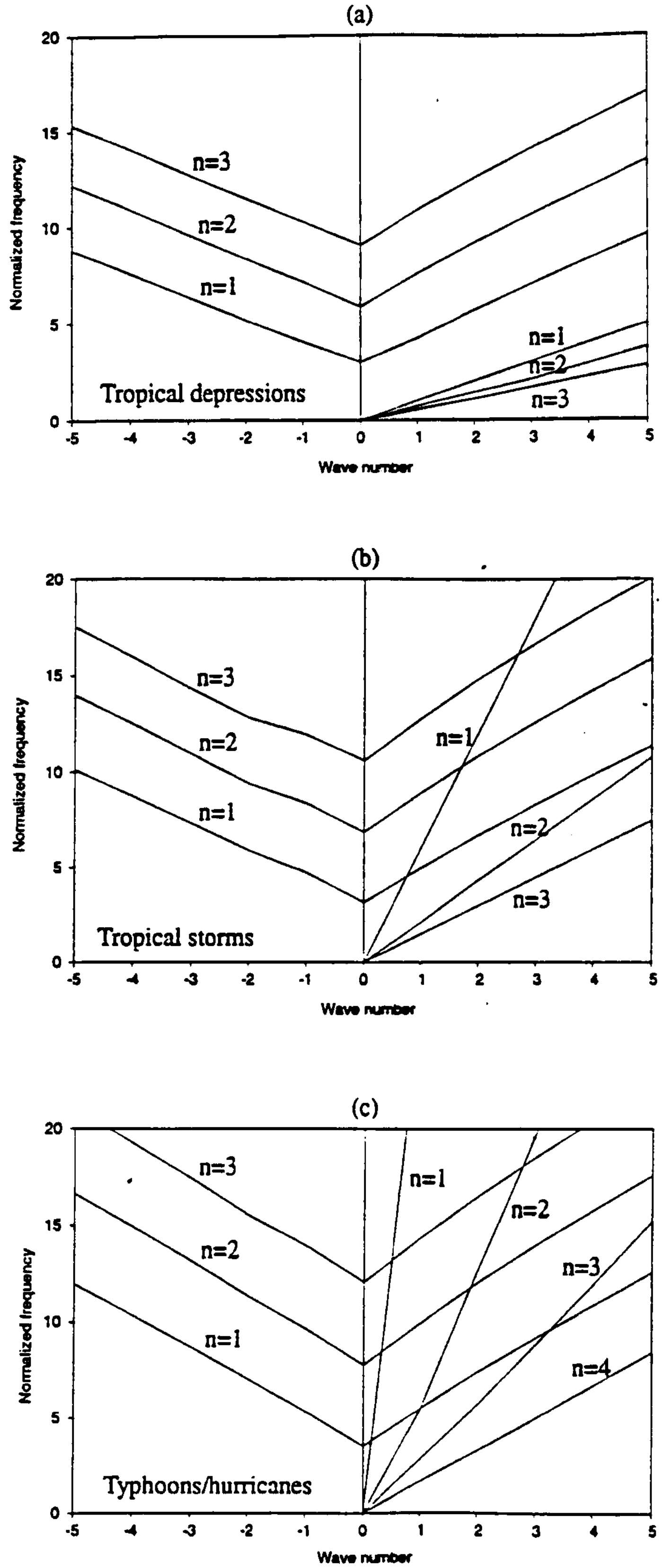


Figure 4.10: Dispersion diagram for both inertia-gravity waves and Rossby waves. (a) Tropical depression, (b) tropical storm, (c) hurricane.

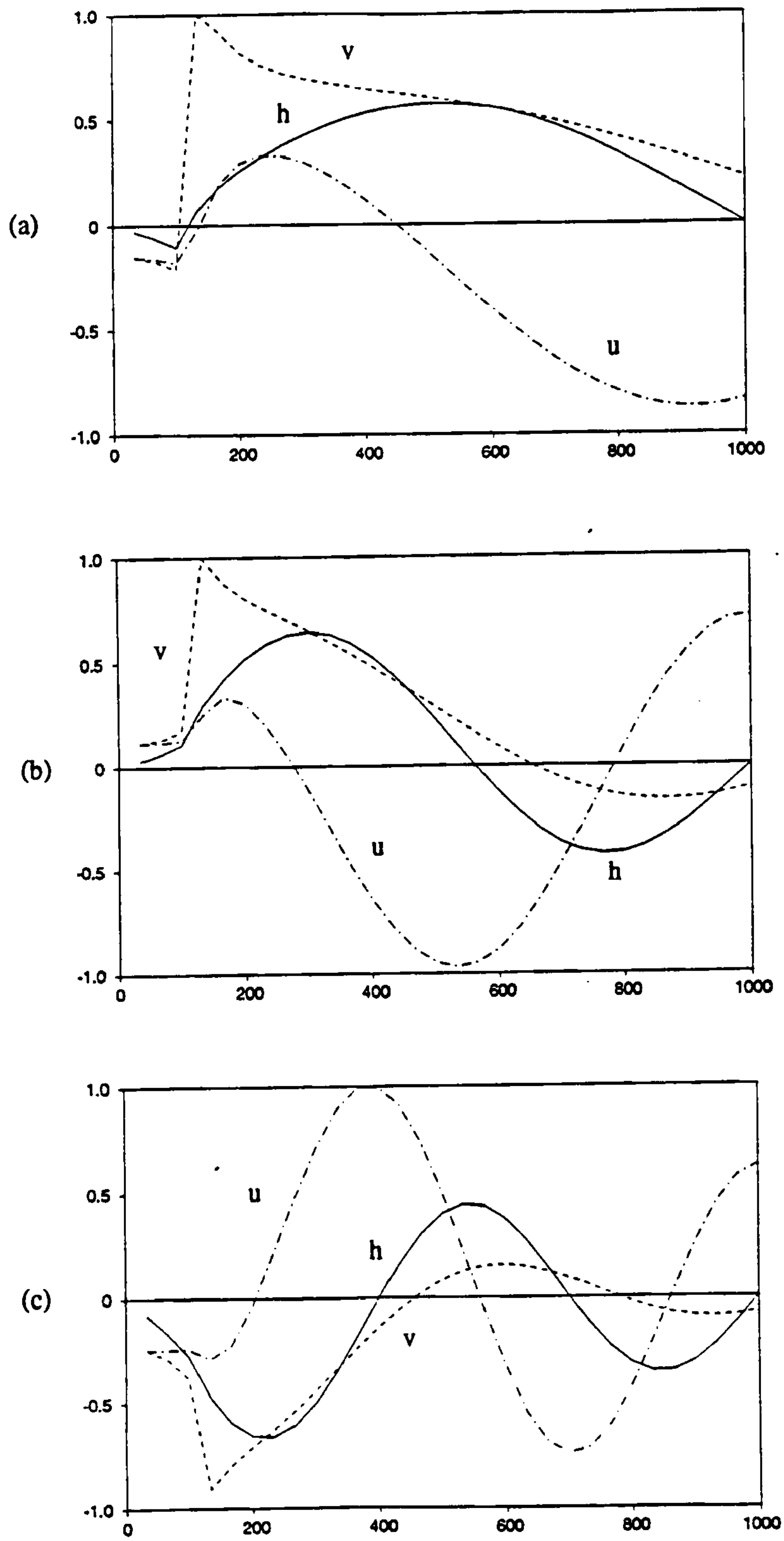


Figure 4.11: Numerically calculated eigenfunctions of inertia-gravity waves for basic vortex of the tropical storm case. (a) For $n = 1$, $s = 1$; (b) for $n = 2$, $s = 1$; (c) for $n = 3$, $s = 1$.

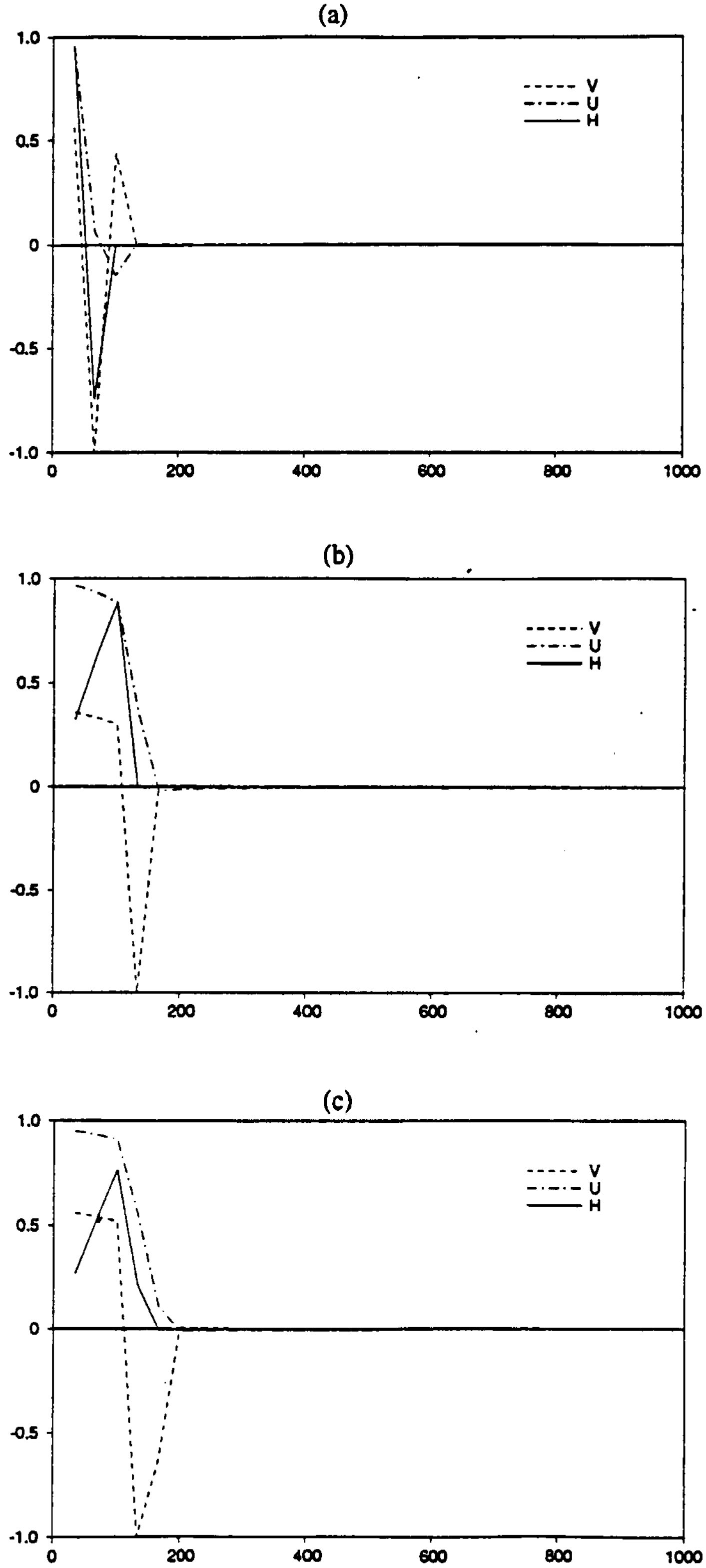


Figure 4.12: Numerically calculated eigenfunctions of Rossby waves for a basic vortex of the tropical storm case. (a) For $n = 1$, $s = 1$; (b) for $n = 2$, $s = 1$; (c) for $n = 3$, $s = 1$.

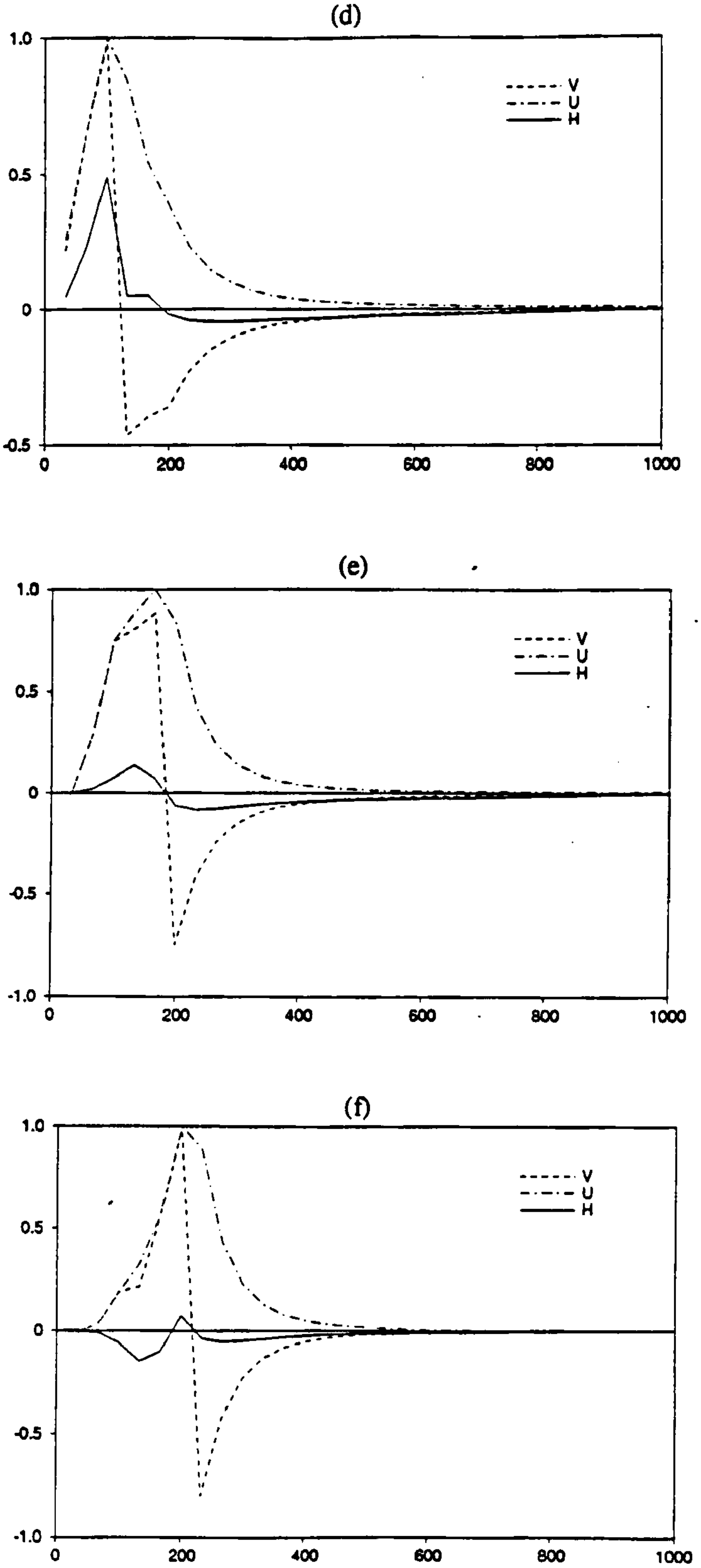


Figure 4.13: Same as Figure 4.12 except (d) for $n = 2$, $s = 2$; (e) for $n = 3$, $s = 3$; (f) for $n = 3$, $s = 1$.

Chapter 5

THE LINEAR DYNAMICS OF MIXED-BALANCE SYSTEMS

The theories developed in Chapters 2 and 3 constitute filtered models, namely the inertia-gravity waves have been removed from the systems. This, however, should be justified in some way. In this chapter we solve the mixed geostrophic-gradient balanced equations, on an f-plane and on a sphere, and compare their eigensolutions to those of the primitive equation models. The comparisons serve two objectives: (1) as we have already stated above, to check the filtering processes; (2) to measure the accuracy of the solutions from these balanced models in comparison with the solutions from the primitive equation models in the linear context.

The chapter is divided into two sections. In section 5.1, the balanced dynamical theory developed in Chapter 2 (the f-plane theory) is applied to the study of linear wave motions in a barotropic circular vortex, in particular a Rankine vortex. Then the results are compared with those from the primitive equation model and the nondivergent barotropic model presented in the previous chapter. A similar analysis of the spherical balanced theory developed in Chapter 3 is conducted in section 5.2, where the perturbations are superimposed on a resting atmosphere. The results are compared with those from tidal theory (Longuet-Higgins, 1968).

5.1 Vorticity (or PV) waves in balanced barotropic vortices

In order to study wave motions in barotropic vortices with balanced signature, we need to find a simple version of the mixed-balance model developed in Chapter 2 with a one-layer vertical structure. Therefore, in the first part of this section we briefly go through the same procedure as presented in Chapters 2 and 3, to derive a shallow water version of the mixed-balance theory. In the second part of this section, we linearize this shallow water balanced model about a basic state of axisymmetry, and obtain a radial structure equation. We then solve this equation by specifying the axisymmetric basic state as Rankine's type, and discuss and compare the solutions with those of the primitive equation model in the final subsection.

5.1.1 The mixed-balance shallow water model

The set of barotropic, balanced governing equations can be derived from the set of shallow water primitive equations through a small Rossby number analysis. The procedures are exactly the same as those presented in Chapters 2 and 3. This leads to the shallow water version of mixed geostrophic-gradient balanced model as follows:

$$\frac{Du_g}{Dt} - \left(f + \frac{v_g}{r}\right)\frac{v}{\gamma} + g\frac{\partial h}{\gamma \partial r} = 0, \tag{5.1}$$

$$\frac{Dv_g}{Dt} + \left(f + \frac{v_g}{r}\right)u + g\frac{\partial h}{r\partial \phi} = 0, \tag{5.2}$$

$$\frac{Dh}{Dt} + h\left(\frac{\partial(ru)}{r\partial r} + \frac{\partial v}{r\partial \phi}\right) = \mathrm{S}, \tag{5.3}$$

where h is the free surface height, S the sources or sinks of h, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + v\frac{\partial}{r\partial \phi} \tag{5.4}$$

is the horizontal material derivative, with the advective processes carried out by the total winds (u, v). Formally similar to semigeostrophic theory, the advected momenta, however, are the geostrophic and gradient winds. They are defined as

$$(fu_g,\ fv_g) = \left(-g\frac{\partial h}{R\partial \phi},\ g\frac{\partial h}{\partial r} - \frac{v_g^2}{r}\right). \tag{5.5}$$

The definition and physical meaning of the distortion factor γ have been discussed in some detail in Chapter 2. Because curvature effects have been considered, and thus the centrifugal force comes into play, this system, (5.1)–(5.5), can describe highly curved flows in a barotropic atmosphere, especially barotropic circular vortices. In the limiting case when the flow has an infinitely small curvature, $\gamma \to 1$, the system reduces to the semigeostrophic equations of uniform vertical structure.

Following the procedure of Chapter 2, we next perform a canonical transformation by using the set of combined geostrophic and potential radius coordinates, i.e., we transform the system from (r, ϕ, t) space to $(R, \Phi, \mathcal{T})$ space, with R and Φ given in (2.55) and (2.56), $t = \mathcal{T}$ being time. Another useful definition is the Bernoulli height:

$$g\mathcal{H} = gh + \tfrac{1}{2}(u_g^2 + v_g^2), \tag{5.6}$$

which can be thought as the free surface height in the transformed space. We can also prove that the derivatives of the physical height and the derivatives of the Bernoulli height are related by

$$\left(\frac{\partial h}{\gamma \partial r}, \frac{\partial h}{\partial \phi}, \frac{\partial h}{\partial t}\right) = \left(\frac{\partial \mathcal{H}}{\partial R} - \frac{u_g^2}{gR}, \frac{\partial \mathcal{H}}{\partial \Phi}, \frac{\partial \mathcal{H}}{\partial \mathcal{T}}\right). \tag{5.7}$$

By the argument presented in Chapter 2, we can neglect the additional term in the first entry on the right hand of (5.7). The discussions and derivations henceforth will not consider this small correction term. With these relations, (5.1) and (5.2) can be transformed into their canonical forms:

$$R\frac{D\Phi}{Dt} = \frac{g}{f}\frac{\partial \mathcal{H}}{\partial R}, \tag{5.8}$$

$$\frac{DR}{Dt} = -\frac{g}{f}\frac{\partial \mathcal{H}}{R\partial \Phi}. \tag{5.9}$$

Substituting these two canonical equations into the total derivative in the transformed space of the form:

$$\frac{D}{Dt} = \frac{\partial}{\partial \mathcal{T}} + U\frac{\partial}{\partial R} + V\frac{\partial}{R\partial \Phi} \tag{5.10}$$

where

$$(U, V) = \left(\frac{DR}{Dt}, R\frac{D\Phi}{Dt}\right), \tag{5.11}$$

we obtain the familiar result that the advective processes in transformed space are now carried out only by the balanced flows. The unbalanced parts of the wind have become implicit in the coordinate transformation.

The vorticity equation (2.66) derived in Chapter 2 is reduced to

$$\frac{D\zeta}{Dt} + \zeta\left(\frac{\partial(ru)}{r\partial r} + \frac{\partial v}{r\partial \phi}\right) = 0, \tag{5.12}$$

where

$$\zeta = f\frac{\partial(\frac{1}{2}R^2, \Phi)}{\partial(\frac{1}{2}r^2, \phi)}. \tag{5.13}$$

The potential vorticity equation can be derived by combining the vorticity equation (5.12) and the continuity equation (5.3). This yields

$$\frac{DP}{Dt} = -P\mathrm{S}, \tag{5.14}$$

where $P = \zeta/h$ is the shallow water potential vorticity. When sources or sinks of surface height are absent, the potential vorticity is a materially conserved quantity, which is the equivalent physical law of Ertel-Rossby's for the shallow water, mixed geostrophic and gradient balanced flows. We now define a new physical quantity, the potential height h^*, in the form

$$h^* = \frac{f}{\zeta}h. \tag{5.15}$$

We see that potential height is smaller than actual height in cyclonic flows, and larger than actual height in anticyclonic flows. In the case of no relative flow, the potential height is the physical height itself. Noting the reciprocal relation between the potential vorticity and potential height, namely $Ph^* = f$, and substituting (5.10), we may finally obtain a predictive equation for potential height:

$$\frac{\partial h^*}{\partial T} + \frac{\partial(RUh^*)}{R\partial R} + \frac{\partial(Vh^*)}{R\partial\Phi} = \mathrm{S}, \tag{5.16}$$

where U and V which have been defined in (5.11) are related to the single variable $\mathcal{H}$. Once we know $\mathcal{H}$ at every time, we may continuously integrate (5.16) forward in time. Therefore, we next need to find an equation for $\mathcal{H}$. We shall be aware of that (5.15) actually gives a diagnostic relation between the potential height h^* and the Bernoulli height $\mathcal{H}$, because (5.15) can be written

$$\frac{\partial(\frac{1}{2}r^2, \phi)}{\partial(\frac{1}{2}R^2, \Phi)}\left[g\mathcal{H} - \tfrac{1}{2}(u_g^2 + v_g^2)\right] = gh^*, \tag{5.17}$$

where $\frac{1}{2}r^2$, ϕ, u_g^2 and v_g^2 can all be expressed in terms of $\mathcal{H}$ through (2.55), (2.56), (5.5) and (5.7). Thus, (5.16) and (5.17) form a closed system. This system is the barotropic version of the mixed-balance theory that we presented in Chapter 2.

5.1.2 Linear dynamics of barotropic mixed-balance theory on an f-plane

We now consider an unforced fluid so that there are no external sources or sinks that affect the free surface. For such a fluid, if we linearize (5.16) about a basic state of rest with a constant height surface, we can only get modes with zero frequencies. These modes correspond to stationary geostrophic modes in the primitive equation solutions discussed in section 4.1.

We next consider a more complicated basic state, a nonresting basic state with a steady, axisymmetric flow, i.e.,

$$(\bar{h}^*,\ \bar{\mathcal{H}}) = (\bar{h}^*(R),\ \bar{\mathcal{H}}(R)). \tag{5.18}$$

From (5.8), (5.9) and (5.11), this basic state gives

$$\bar{U} = 0, \quad \bar{V} = \frac{g}{f}\frac{\partial \bar{\mathcal{H}}}{\partial R}. \tag{5.19}$$

After we linearize the potential height equation (5.16) about the basic state (5.18) and (5.19), we obtain

$$\frac{\partial h^{*\prime}}{\partial T} + \bar{V}\frac{\partial h^{*\prime}}{R\partial \Phi} - \frac{g}{f}\frac{\partial \bar{h}^*}{\partial R}\frac{\partial \mathcal{H}'}{R\partial \Phi} = 0. \tag{5.20}$$

Although $\bar{V}$ and $\bar{h}^*$ are given by the structure of the basic flow, (5.20) is still not closed. In order to solve this equation, we have to also linearize the invertibility principle to find the relationship between $\mathcal{H}'$ and $h^{*\prime}$. From (5.5) and (5.7), we note that

$$(fu_g,\ fv_g) = g\left(-\frac{\partial \mathcal{H}}{R\partial \Phi},\ \frac{r}{R}\frac{\partial \mathcal{H}}{\partial R}\right), \tag{5.21}$$

which gives

$$\bar{u}_g = 0, \quad u_g' = -\frac{g}{f}\frac{\partial \mathcal{H}'}{R\partial \Phi}, \tag{5.22}$$

$$\bar{v}_g = \frac{g}{f}\frac{\bar{r}}{R}\frac{\partial \bar{\mathcal{H}}}{\partial R} = \bar{v}_g(R), \tag{5.23}$$

and

$$v_g' = \frac{g}{f}\frac{\bar{r}}{R}\frac{\partial \mathcal{H}'}{\partial R} + \frac{r'}{\bar{r}}\bar{v}_g. \tag{5.24}$$

The second term on the right-hand side of (5.24) can be neglected in most physical situations, because it is often true that $r'/\bar{r} \ll v_g'/\bar{v}_g$. In (5.23), we have also used the fact that

$\bar{r} = \bar{r}(R)$, which we shall prove next. Considering the geostrophic and potential radius coordinates (2.55) and (2.56), we have

$$\bar{\phi} = \Phi, \quad \phi' = \phi - \bar{\phi} = \frac{u'_g}{fR}, \tag{5.25}$$

and

$$\tfrac{1}{2}\bar{r}^2 = \frac{\frac{1}{2}R^2}{1 + \frac{2g}{f^2 R}\frac{\partial \bar{\mathcal{H}}}{\partial R}} = \bar{\Gamma}(R),$$

$$\tfrac{1}{2}r^{2\prime} = -\frac{\frac{gR}{f^2}\frac{\partial \mathcal{H}'}{\partial R}}{\left(1 + \frac{2g}{f^2 R}\frac{\partial \bar{\mathcal{H}}}{\partial R}\right)^2} = -\frac{4g\bar{\Gamma}^2}{f^2 R^3}\frac{\partial \mathcal{H}'}{\partial R}. \tag{5.26}$$

With these preliminary steps, we can now linearize the invertibility equation (5.17) to obtain

$$g(\bar{h}^* + h^{*\prime}) = \frac{\partial(\frac{1}{2}\bar{r}^2 + \frac{1}{2}r^{2\prime}, \bar{\phi} + \phi')}{\partial(\frac{1}{2}R^2, \Phi)}\left[g(\bar{\mathcal{H}} + \mathcal{H}') - \tfrac{1}{2}(\bar{u}_g + u'_g)^2 - \tfrac{1}{2}(\bar{v}_g + v'_g)^2\right]. \tag{5.27}$$

Note that the basic state should exactly satisfy the governing equations so that

$$g\bar{h}^* = \frac{\partial(\frac{1}{2}\bar{r}^2, \bar{\phi})}{\partial(\frac{1}{2}R^2, \Phi)}\left[g\bar{\mathcal{H}} - \tfrac{1}{2}\bar{v}_g^2\right]. \tag{5.28}$$

Equation (5.27) then becomes

$$\begin{aligned} gh^{*\prime} &= \frac{\partial(\frac{1}{2}\bar{r}^2)}{R\partial R}(g\mathcal{H}' - \bar{v}_g v'_g) + \frac{\partial(\frac{1}{2}r^{2\prime})}{R\partial R}(g\bar{\mathcal{H}} - \tfrac{1}{2}\bar{v}_g^2) + \frac{\partial(\frac{1}{2}\bar{r}^2)}{R\partial R}\frac{\partial \phi'}{\partial \Phi}(g\bar{\mathcal{H}} - \tfrac{1}{2}\bar{v}_g^2) \\ &= \frac{\partial \bar{\Gamma}}{R\partial R}(g\mathcal{H}' - \bar{v}_g v'_g) - g\bar{h}\frac{\partial}{R\partial R}\left(\frac{4g\bar{\Gamma}^2}{f^2 R^3}\frac{\partial \mathcal{H}'}{\partial R}\right) + g\bar{h}\frac{\partial \bar{\Gamma}}{R\partial R}\frac{\partial}{\partial \Phi}\left(\frac{u'_g}{fR}\right). \end{aligned}$$

On substituting in (5.22) and (5.24), and denoting $\Upsilon = \partial\bar{\Gamma}/R\partial R$, we finally obtain the linearized invertibility principle

$$h^{*\prime} = \Upsilon \mathcal{H}' - \frac{\Upsilon \bar{r}\bar{v}_g}{f}\frac{\partial \mathcal{H}'}{R\partial R} - \frac{4g\bar{h}}{f^2}\frac{\partial}{R\partial R}\left(\frac{\bar{\Gamma}^2}{R^3}\frac{\partial \mathcal{H}'}{\partial R}\right) - \frac{\Upsilon g\bar{h}}{f^2 R^2}\frac{\partial^2 \mathcal{H}'}{\partial \Phi^2}. \tag{5.29}$$

Let us now substitute (5.29) into (5.20), and assume the perturbation Bernoulli height to have the following wave form

$$\mathcal{H}'(R, \Phi, T) = \hat{\mathcal{H}}(R)e^{i(s\Phi - \sigma T)}. \tag{5.30}$$

After these substitutions, we obtain the radial structure equation in the form

$$\frac{4g\bar{h}}{f^2}\frac{d}{RdR}\left(\frac{\bar{\Gamma}^2}{R^3}\frac{d\hat{\mathcal{H}}}{dR}\right)+\frac{\Upsilon\bar{r}\bar{v}_g}{f}\frac{d\hat{\mathcal{H}}}{RdR}-\left[\frac{s^2\Upsilon g\bar{h}}{f^2R^2}+\frac{s}{f\hat{\sigma}}\frac{d(g\bar{h}^*)}{RdR}+\Upsilon\right]\hat{\mathcal{H}}=0, \tag{5.31}$$

where $\hat{\sigma}=\sigma-\bar{V}s/R$ is the Doppler-shifting frequency. From the definitions of the potential height, (5.15), and the gradient wind, (5.23), we also note that

$$g\bar{h}^*=\frac{f}{\zeta}g\bar{h}=\Upsilon g\bar{h}, \tag{5.32}$$

and

$$\bar{r}\bar{v}_g=\frac{\bar{r}^2}{R}\bar{V}=\frac{2\bar{\Gamma}\bar{V}}{R}. \tag{5.33}$$

With all these results, we can rewrite (5.31)

$$\frac{d}{dR}\left(\frac{\bar{\Gamma}^2}{R^3}\frac{d\hat{\mathcal{H}}}{dR}\right)+\frac{f\Upsilon\bar{\Gamma}\bar{V}}{2Rg\bar{h}}\frac{d\hat{\mathcal{H}}}{dR}-\left[\frac{s^2\Upsilon}{4R}+\frac{fs}{4\hat{\sigma}\bar{h}}\frac{d(\Upsilon\bar{h})}{dR}+\frac{f^2R\Upsilon}{4g\bar{h}}\right]\hat{\mathcal{H}}=0, \tag{5.34}$$

This equation presents an eigenvalue problem. There is, obviously, only one class of eigenfrequencies in this system. In contrast with the primitive equation model, the shallow water primitive equation system presented in the previous chapter for instance, where three classes waves are found, the present balanced system, (5.16) and (5.17) or its linear form (5.34), constitutes a filtered model. One should also compare (5.34) with the nondivergent barotropic model discussed in Chapter 4, equation (4.38) say. Equation (5.34) exactly reduces to (4.38) under two conditions: (1) the curvature of the fluid is infinitely small, i.e., $R_{oc}=V/fr\to 0$, so that $R\simeq r$, and $\bar{\Gamma}\simeq\frac{1}{2}R^2$, $\Upsilon\simeq 1$; (2) the flow is in a nondivergent limit, i.e., the mean depth of the fluid $g\bar{h}\to\infty$. Under these circumstances, (5.34) becomes

$$R\frac{d}{dR}\left(R\frac{d\hat{\mathcal{H}}}{dR}\right)-s^2\hat{\mathcal{H}}=0, \tag{5.35}$$

which is exactly the Kelvin problem discussed in Chapter 4. This also proves that our mixed-balance theory is more general than the quasi-geostrophic and semigeostrophic theories, because QG and SG generalize the nondivergent barotropic system by including the proper vertical structure and the divergence of the fluids. This, by the analysis above, is just one aspect of the generalization. Our mixed-balance theory not only includes the proper vertical structure (see Chapters 2 and 3), and divergence of the fluids, but also allows curvature of the flows.

5.1.3 The potential vorticity waves associated with Rankine's vortex

Equation (5.34) is a linear, second order differential equation with variable coefficients. For a resting basic state, there only exist stationary modes as we pointed out earlier. For any nonresting basic state, the problem is so complicated that the analytical solution method is impossible. Therefore, we now attempt to solve (5.34) numerically. Because the Doppler-shifted frequency is a function of radius, we have to reformulate (5.34) into a standard eigenvalue problem. Thus, we write (5.34) in the form

$$\mathcal{A}(\hat{\mathcal{H}}) = \sigma\mathcal{B}(\hat{\mathcal{H}}), \tag{5.36a}$$

where $\mathcal{A}$ and $\mathcal{B}$ are linear operators defined as

$$\mathcal{A}(\,) = \bar{a}\frac{d}{dR}\left(\bar{\alpha}\frac{d(\,)}{dR}\right) + \bar{b}\frac{d(\,)}{dR} - \bar{c}(\,), \tag{5.36b}$$

$$\mathcal{B}(\,) = \bar{d}\frac{d}{dR}\left(\bar{\alpha}\frac{d(\,)}{dR}\right) + \bar{e}\frac{d(\,)}{dR} - \bar{f}(\,), \tag{5.36c}$$

where $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$, $\bar{f}$ and $\bar{\alpha}$ are all variable coefficients which relate to the nonresting basic state of axisymmetry. They are:

$$\bar{a} = 4Rs\bar{V}g\bar{h}, \quad \bar{b} = 2fs\Upsilon\bar{\Gamma}\bar{V}^2, \quad \bar{c} = s^3\bar{V}\Upsilon g\bar{h} - fsgR^2\frac{d(\Upsilon\bar{h})}{dR} + f^2R^2s\bar{V}\Upsilon,$$

$$\bar{d} = 4R^2g\bar{h}, \quad \bar{e} = 2fR\Upsilon\bar{\Gamma}\bar{V}, \quad \bar{f} = s^2R\Upsilon g\bar{h} + f^2R^3\Upsilon, \quad \bar{\alpha} = \frac{\bar{\Gamma}^2}{R^3}.$$

Equation (5.36a) is discretized on a uniform grid in the radial direction, i.e., $R = n\Delta R$, where $n = 1, 2, 3, \ldots$, and ΔR is the grid spacing. We then solve the eigenvalue problem of the form

$$\mathbf{AX} = \sigma\mathbf{BX}, \tag{5.37}$$

where σ is the eigenvalue, $\mathbf{A}$ and $\mathbf{B}$ are the coefficient matrices, and $\mathbf{X}$ are the corresponding eigenvectors.

We next specify the basic state as Rankine's vortex, which we have studied in great a detail in the previous chapter. The tangential wind as the function of R takes the form

$$\bar{V} = \begin{cases} \frac{1}{2}\xi R & 0 \le R \le R_1 \\ \frac{1}{2}\xi\frac{R_1^2}{R} & R_1 \le R \le \infty \end{cases}, \tag{5.38}$$

or it can be expressed in terms of its streamfunction as

$$g\bar{\mathcal{H}} = \begin{cases} g\mathcal{H}_0 + \frac{1}{4}f\xi(R^2 - R_1^2) & 0 \le R \le R_1 \\ g\mathcal{H}_0 + \frac{1}{2}f\xi R_1^2 \ln\left(\frac{R}{R_1}\right) & R_1 \le R \le \infty \end{cases}, \tag{5.39}$$

where ξ and $\mathcal{H}_0$ are constants, R_1 is the radius of inner core or the radius at which the tangential wind reaches its maximum value in Rankine's vortex. From (5.26), we find

$$\bar{\Gamma} = \begin{cases} \frac{\frac{1}{2}R^2}{1+\frac{\xi}{f}} & 0 \le R \le R_1 \\ \frac{\frac{1}{2}R^2}{1+\frac{\xi}{f}\left(\frac{R_1}{R}\right)^2} & R_1 \le R \le \infty \end{cases}. \tag{5.40}$$

As defined previously, Υ is just the derivative of $\bar{\Gamma}$, thus

$$\Upsilon = \begin{cases} \frac{1}{1+\frac{\xi}{f}} & 0 \le R \le R_1 \\ \frac{1+\frac{2\xi}{f}\left(\frac{R_1}{R}\right)^2}{\left[1+\frac{\xi}{f}\left(\frac{R_1}{R}\right)^2\right]^2} & R_1 \le R \le \infty \end{cases}. \tag{5.41}$$

The derivative of Υ with respect to R, which appears in the coefficient $\bar{c}$, can be found using

$$\frac{d\Upsilon}{dR} = \begin{cases} 0 & 0 \le R \le R_1 \\ \frac{4\frac{\xi^2}{f^2}\left(\frac{R_1}{R}\right)^4}{R\left[1+\frac{\xi}{f}\left(\frac{R_1}{R}\right)^2\right]^3} & R_1 \le R \le \infty \end{cases}. \tag{5.42}$$

From (5.6), (5.23) and (5.26), we can express the physical height in terms of quantities in transformed space

$$g\bar{h} = g\bar{\mathcal{H}} - \tfrac{1}{2}\bar{v}_g^2 = g\bar{\mathcal{H}} - \frac{\bar{\Gamma}\bar{V}^2}{R^2}, \tag{5.43}$$

which gives

$$g\bar{h} = \begin{cases} g\mathcal{H}_0 + \frac{1}{4}f\xi(R^2 - R_1^2) - \frac{\frac{1}{8}\xi^2R^2}{1+\frac{\xi}{f}} & 0 \le R \le R_1 \\ g\mathcal{H}_0 + \frac{1}{2}f\xi R_1^2 \ln\left(\frac{R}{R_1}\right) - \frac{\frac{1}{8}\xi^2R_1^4}{R^2+\frac{\xi}{f}R_1^2} & R_1 \le R \le \infty \end{cases}, \tag{5.44}$$

and its derivative with respect to R

$$\frac{d(g\bar{h})}{dR} = \begin{cases} \frac{1}{2}f\xi R - \frac{\frac{1}{4}\xi^2R}{1+\frac{\xi}{f}} & 0 \le R \le R_1 \\ \frac{1}{2}f\xi\frac{R_1^2}{R} + \frac{\frac{1}{4}\xi^2R_1^4R}{\left[R^2+\frac{\xi}{f}R_1^2\right]^2} & R_1 \le R \le \infty \end{cases}. \tag{5.45}$$

Equations (5.38)–(5.45) are used in (5.36) or (5.37). The intensity ξ and the size R_1 of the Rankine vortex are chosen according to the values listed in Table 4.2. The eigenfrequencies computed from the mixed-balance model are compared with those from the primitive equation model (see Chapter 4) side by side in Tables 5.1–5.5 for azimuthal wavenumbers 1 to 5. The numbers listed in each Table are the frequencies normalized by the coriolis parameter, and each column contains the first 10 radial modes.

From these tables we can apparently identify that the eigenmodes predicted by the mixed-balance model are of the Rossby type, and they compare remarkably well to the eigensolutions from the primitive equation model. These points can also be illustrated by the dispersion diagrams, shown in Figure 5.1. The three panels in this figure are for three Rankine vortices of different intensities, with increasing intensity from top to bottom. The solid lines are the dispersion curves calculated by the mixed-balance model, and the dashed lines are those by the primitive equation model, for five different radial modes. The solutions from two models are consistent. In complementing the discussion of (5.34) and (5.35), we also plot the solutions from the nondivergent barotropic model (dotted lines). The single frequency curve and its shift along the wavenumber axis are, again, the two main features presented by this nondivergent barotropic system. These features, however, are seen neither in the primitive equation model nor in the mixed-balance model, which may, as we discussed in the previous chapter, be due to the differences of the vorticity model and the potential vorticity models; of the nondivergent flow system and the divergent flow systems.

The balanced winds can be computed from the eigenfunction $\mathcal{H}$ by (5.8), (5.9) and (5.11). Figure 5.2 shows these balanced mass and winds corresponded to eigenfrequencies $s = 1$, for $n = 1, 2, 3$ of the Rankine vortex with size and intensity corresponding to tropical storm, from the top to bottom panels. Figure 5.3 is the similar plot for $s = 2$, $n = 2$; $s = 3$, $n = 3$; and $s = 4$, $n = 4$ modes. These eigenvector plots resemble those of Rossby wave motions from the primitive equation model and the nondivergent barotropic model (see Figures 4.8, 4.12 and 4.13 of Chapter4), which further indicates that our mixed-balance theory does filter the inertia-gravity waves, and retains reasonable accuracy of the original governing laws of the primitive equations. However, there are still some differences in eigenfunctions for larger radial modes from the two model simulations.

$s = 1$						
	Tropical depressions		Tropical storms		Typhoons/Hurricanes	
	PE	Balanced	PE	Balanced	PE	Balanced
n	1.001	1.003	5.976	6.009	–	27.051
	0.722	0.688	3.136	3.850	–	5.465
	0.556	0.598	2.074	2.347	2.843	3.468
	0.441	0.467	1.462	1.596	1.646	1.833
	0.358	0.375	1.084	1.157	1.058	1.146
	0.296	0.308	0.834	0.878	0.738	0.785
	0.249	0.257	0.661	0.690	0.544	0.572
	0.212	0.219	0.537	0.556	0.417	0.436
	0.184	0.188	0.444	0.458	0.331	0.343
	0.160	0.164	0.374	0.384	0.268	0.277

Table 5.1: The eigenfrequencies computed from the mixed-balance model are compared with those from the shallow water primitive equation model for azimuthal wavenumber 1. Each column presents the first 10 n modes.

$s = 2$						
	Tropical depressions		Tropical storms		Typhoons/Hurricanes	
	PE	Balanced	PE	Balanced	PE	Balanced
n	2.004	2.005	11.967	12.028	–	54.126
	1.438	1.406	5.880	6.334	15.459	12.707
	1.101	1.081	4.307	4.070	5.735	5.503
	0.886	0.858	2.964	2.844	3.293	3.114
	0.716	0.698	2.181	2.102	2.123	2.018
	0.592	0.579	1.673	1.618	1.481	1.417
	0.497	0.489	1.324	1.285	1.092	1.051
	0.424	0.418	1.074	1.045	0.838	0.810
	0.366	0.361	0.888	0.867	0.663	0.644
	0.319	0.315	0.747	0.731	0.537	0.525

Table 5.2: Same as Table 5.1 except for azimuthal wavenumber 2.

	$s=3$					
	Tropical depressions		Tropical storms		Typhoons/Hurricanes	
	PE	Balanced	PE	Balanced	PE	Balanced
	3.006	3.007	17.976	18.043	–	81.195
	2.087	2.139	10.388	9.770	22.686	19.719
	1.688	1.644	6.413	6.248	8.815	8.601
	1.327	1.304	4.457	4.351	4.989	4.823
n	1.075	1.060	3.280	3.208	3.203	3.104
	0.889	0.878	2.515	2.465	2.230	2.170
	0.747	0.740	1.990	1.954	1.642	1.604
	0.637	0.632	1.613	1.587	1.259	1.235
	0.549	0.546	1.334	1.315	0.996	0.980
	0.479	0.476	1.122	1.107	0.807	0.796

Table 5.3: Same as Table 5.1 except for azimuthal wavenumber 3.

	$s=4$					
	Tropical depressions		Tropical storms		Typhoons/Hurricanes	
	PE	Balanced	PE	Balanced	PE	Balanced
	4.008	4.010	23.994	24.058	108.779	108.263
	2.909	2.876	14.864	13.206	29.907	26.622
	2.245	2.209	8.577	8.432	11.985	11.683
	1.771	1.750	5.961	5.860	6.707	6.539
n	1.435	1.421	4.384	4.315	4.291	4.193
	1.187	1.177	3.360	3.311	2.982	2.924
	0.998	0.991	2.657	2.622	2.193	2.158
	0.850	0.845	2.153	2.129	1.681	1.658
	0.733	0.730	1.781	1.763	1.329	1.314
	0.639	0.636	1.497	1.483	1.077	1.067

Table 5.4: Same as Table 5.1 except for azimuthal wavenumber 4.

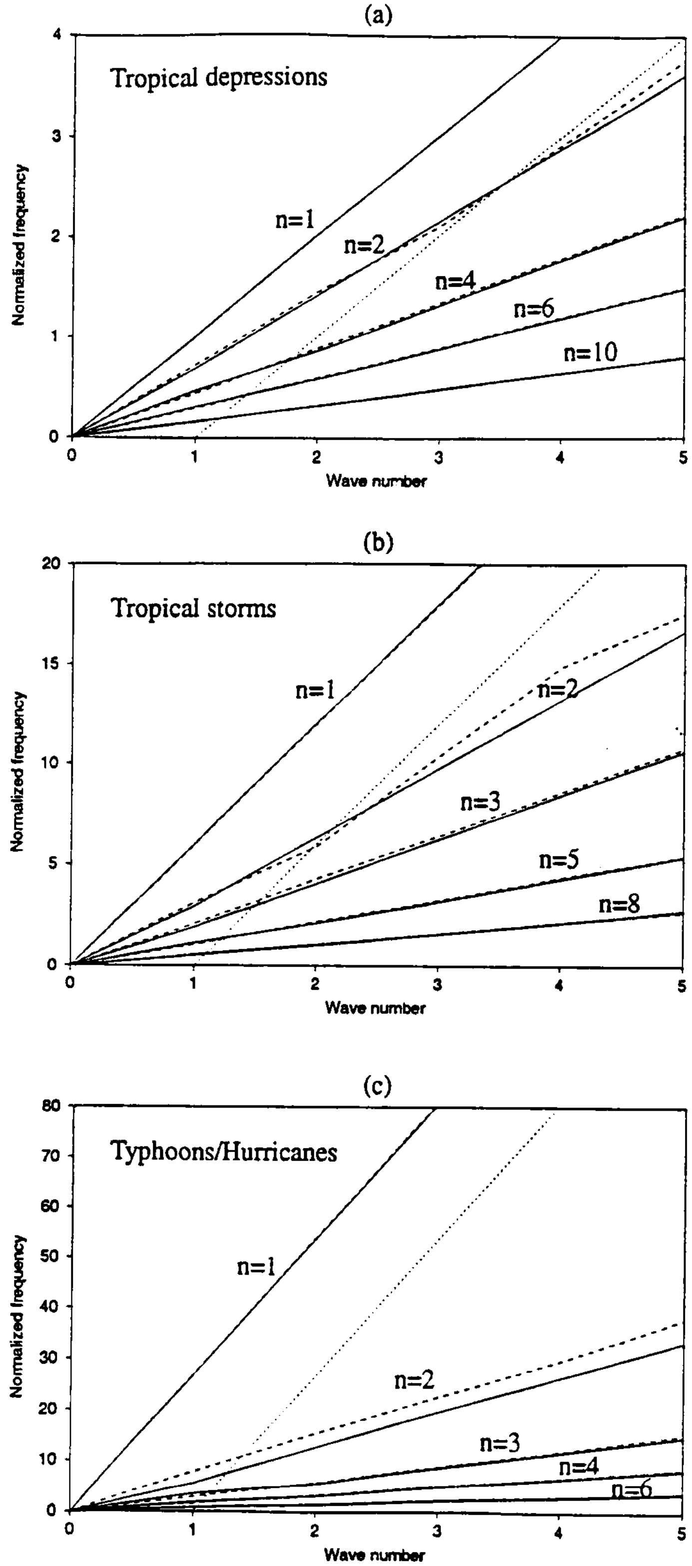


Figure 5.1: Dispersion diagrams compares Rossby wave frequencies from the mixedbalance models (solid curves), the primitive equation model (dashed curves) and the nondivergent barotropic model (dotted curves). From the top to bottom panels are for vortices with different intensities.

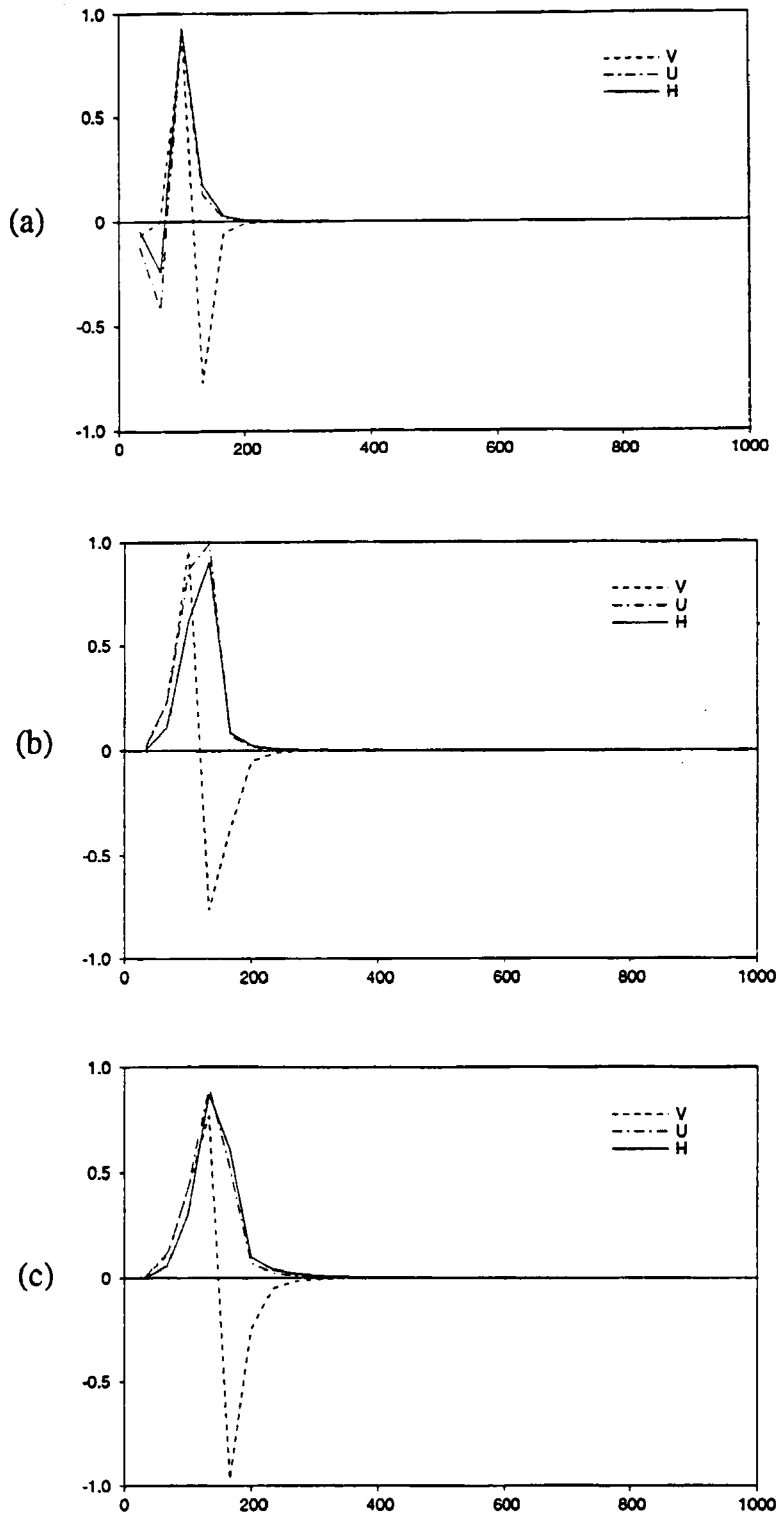
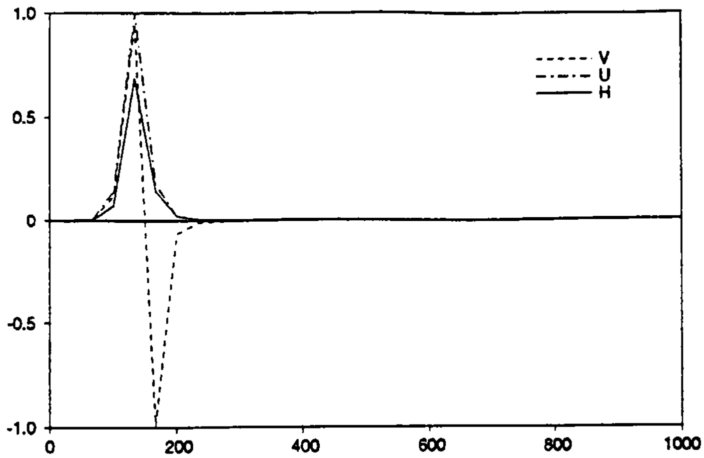
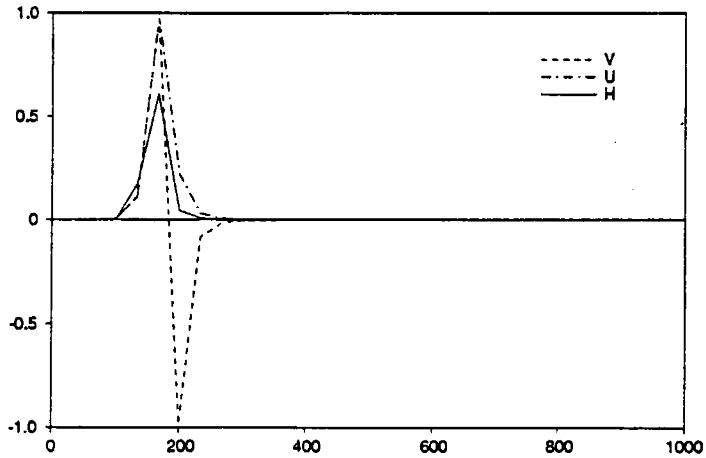


Figure 5.2: Balanced wind and mass fields calculated by the mixed-balance model. (a) For $n = 1$, $s = 1$; (b) for $n = 2$, $s = 1$; (c) for $n = 3$, $s = 1$.

(d)



(e)



(f)

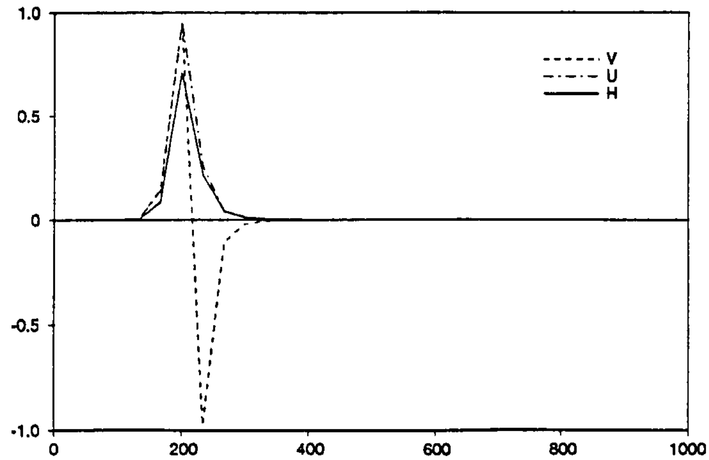


Figure 5.3: Same as Figure 5.2 except (d) For $n = 2$, $s = 2$; (e) for $n = 3$, $s = 3$; (f) for $n = 4$, $s = 4$.

$s = 5$						
	Tropical depressions		Tropical storms		Typhoons/Hurricanes	
	PE	Balanced	PE	Balanced	PE	Balanced
	5.009	5.012	30.014	30.073	126.342	135.330
	3.761	3.613	17.499	16.633	38.023	33.474
	2.807	2.773	10.761	10.617	15.239	14.745
	2.216	2.196	7.471	7.372	8.445	8.254
n	1.796	1.782	5.491	5.422	5.385	5.283
	1.485	1.475	4.206	4.158	3.738	3.679
	1.248	1.241	3.325	3.291	2.747	2.711
	1.063	1.059	2.694	2.670	2.104	2.082
	0.917	0.914	2.227	2.210	1.663	1.649
	0.799	0.797	1.872	1.860	1.348	1.338

Table 5.5: Same as Table 5.1 except for azimuthal wavenumber 5.

5.2 The linear eigenmodes of the mixed-balance model on a sphere

In this section, we first briefly review the tidal wave theory and present some of the solutions obtained by Longuet-Higgins (1968). We then discuss the linear dynamics of the mixed-balance theory on sphere, comparing solutions of this model with those of the tidal theory.

5.2.1 The eigenfrequencies and eigenfunctions of Laplace's tidal equation

Let us now take a set of shallow water version of (3.6)–(3.9), in Chapter 3, and linearize this set about a basic state of rest with a constant height surface $(\bar{u}, \bar{v}, \bar{h}) = (0, 0, H)$. This leads to Laplace's tidal equations

$$\frac{\partial u'}{\partial t} - 2\Omega v' \sin\phi + g\frac{\partial h'}{a\cos\phi\partial\lambda} = 0, \tag{5.46}$$

$$\frac{\partial v'}{\partial t} + 2\Omega u' \sin\phi + g\frac{\partial h'}{a\partial\phi} = 0, \tag{5.47}$$

$$\frac{\partial h'}{\partial t} + H\left[\frac{\partial u'}{a\cos\phi\partial\lambda} + \frac{\partial(v'\cos\phi)}{a\cos\phi\partial\phi}\right] = 0, \tag{5.48}$$

where u', v' and h' are assumed to have wave solutions in the form:

$$\begin{pmatrix} u' \\ v' \\ h' \end{pmatrix} = \begin{pmatrix} \hat{u}(\phi) \\ \hat{v}(\phi) \\ \hat{h}(\phi) \end{pmatrix} e^{i(s\lambda - \sigma t)}. \tag{5.49}$$

Here s is the zonal wavenumber and σ is the frequency. On substituting (5.49) into (5.46)–(5.48), and solving the first two equations, we obtain

$$\hat{u} = \frac{g}{2\Omega(\omega^2 - \sin^2\phi)} \left(\frac{\omega s}{a\cos\phi} + \sin\phi \frac{\partial}{a\partial\phi} \right) \hat{h}, \tag{5.50}$$

$$\hat{v} = \frac{-ig}{2\Omega(\omega^2 - \sin^2\phi)} \left(\frac{s\tan\phi}{a} + \omega \frac{\partial}{a\partial\phi} \right) \hat{h}, \tag{5.51}$$

where $\omega = \sigma/2\Omega$ is the normalized frequency. On substituting (5.50) and (5.51) into (5.48), the Laplace tidal equations can be combined into one differential equation with one variable:

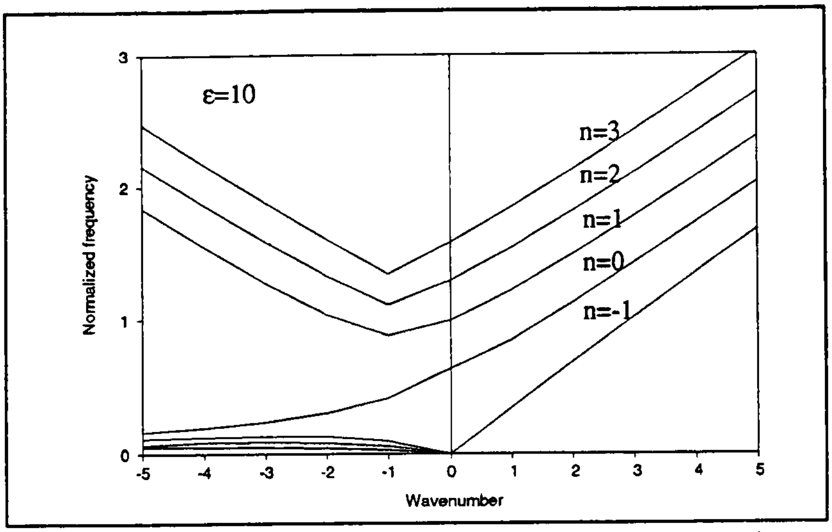
$$\mathcal{L}(\hat{h}) = \epsilon\hat{h}, \tag{5.52}$$

where $\epsilon = 4\Omega^2 a^2/gH$ is Lamb's parameter, and $\mathcal{L}$ denotes the linear operator

$$\mathcal{L} = \frac{d}{\cos\phi d\phi}\left(\frac{\cos\phi}{\sin^2\phi - \omega^2} \frac{d}{d\phi} \right) + \frac{1}{\sin^2\phi - \omega^2} \left(\frac{s(\sin^2\phi + \omega^2)}{\omega(\sin^2\phi - \omega^2)} - \frac{s^2}{\cos^2\phi} \right). \tag{5.53}$$

From (5.53), one can see that (5.52) presents a cubic equation in ω. Thus, it is deducible that there are three classes of wave solutions predicted by the Laplace tidal equations. These solutions are traditionally referred to as the westward propagating, and eastward propagating waves of the first class, and the westward propagating waves of the second class (Margules, 1893; Hough, 1898; Haurwitz, 1940; Dikii, 1966; Longuet-Higgins, 1968). Longuet-Higgins (1968) performed a numerical calculation of (5.52) for several different choices of Lamb's parameter. In Figure 5.4, we plot the dispersion diagrams for $\epsilon = 10$ and $\epsilon = 1000$ according to the data listed in Table 5 of Longuet-Higgins (1968). For the purpose of comparison with the results of the mixed-balance model, we also adopt the eigenvector plots of Longuet-Higgins' corresponding to the eigenvalues of $s = 1, 2$ for $n - s = 2$ mode of wave class 2 (shown in Figure 5.5).

(a)



(b)

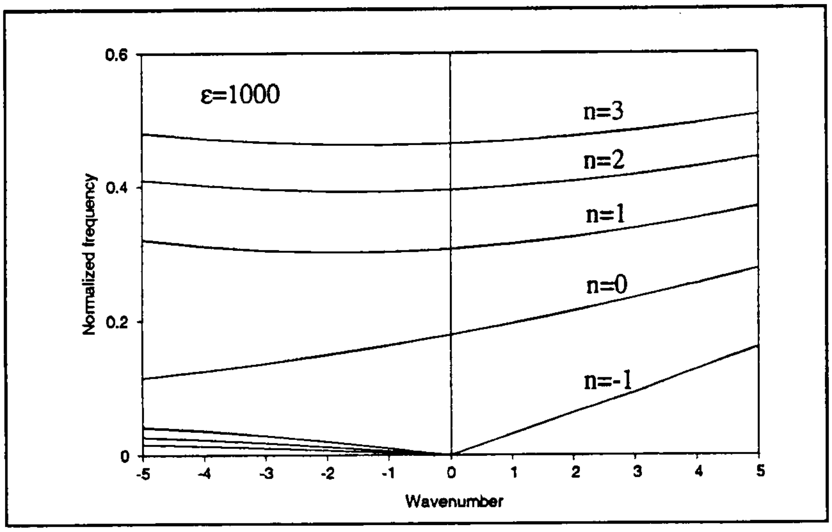


Figure 5.4: Dispersion diagrams from the Laplace tidal equations. The two panels are for two different choices of Lamb's parameter. Plotted according to numerical data by Longuet-Higgins (1968).

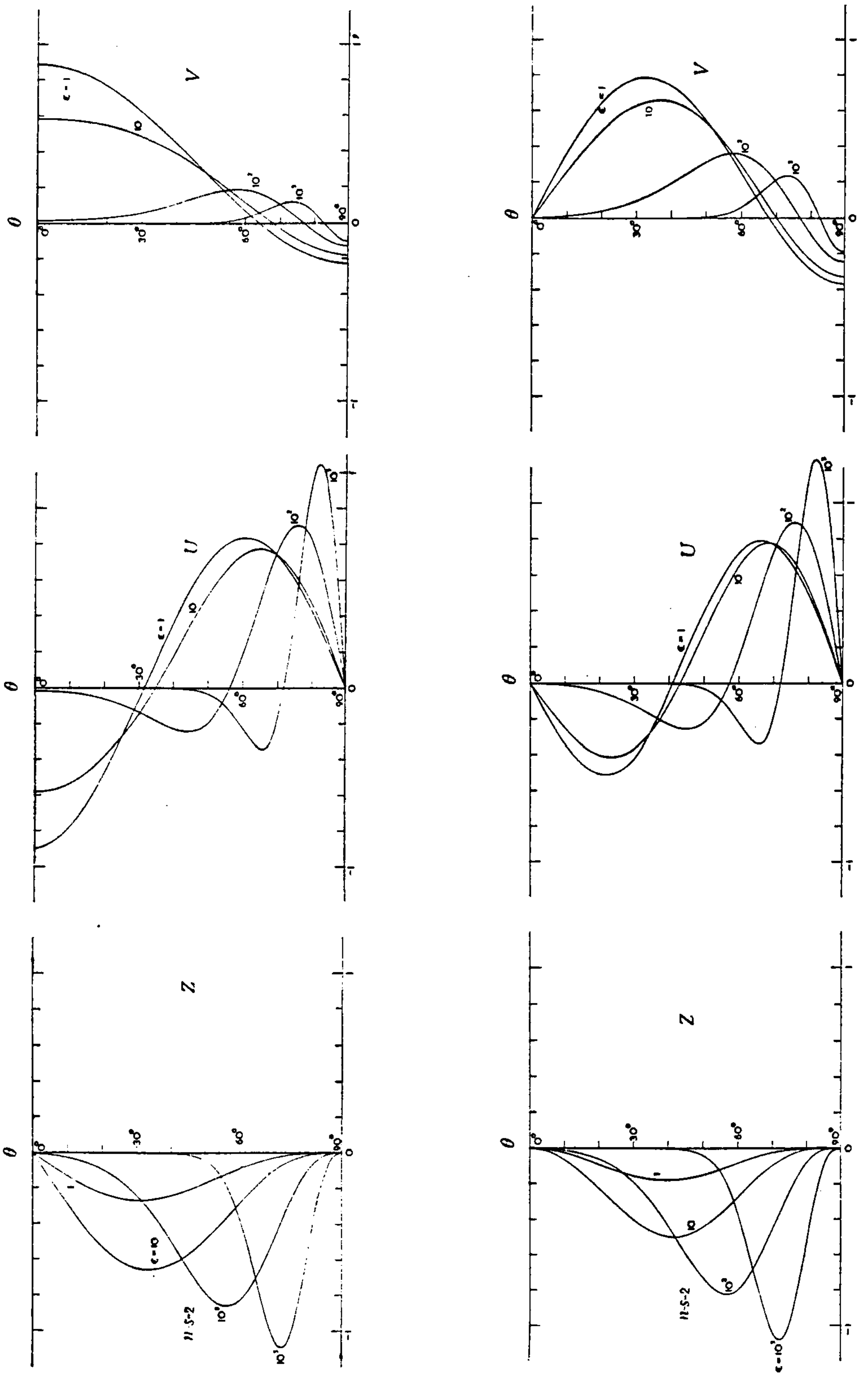


Figure 5.5: Eigenfunctions for waves of the second class from the Laplace tidal equations. Left column is for $s = 1$, and the right for $s = 2$.

5.2.2 The linear eigenmodes of the mixed-balance model on a sphere

We now begin to consider a mix-balanced dynamical system on the sphere that has been summarized in Table 3.1. In order that the eigensolutions from this mixed-balance model are comparable to those of the shallow water primitive equation model discussed in the previous section, we assume an adiabatic flow motion $\dot{s} = 0$, and a basic state of rest with Boussinesq density profiles both in physical and pseudo-physical spaces. This basic state is summarized below:

$$\left\{ \begin{array}{c} \bar{\rho} = \rho_0 = const. \\ \bar{\sigma}^* = \sigma_0 = \frac{p_B - p_T}{S_T - S_B} = const. \\ \bar{u}_g = 0, \quad \bar{v}_g = 0 \end{array} \right\} \tag{5.54}$$

where the bar quantities denotes the quantities at basic state, ρ is the density in physical space. In a resting basic state, the potential pseudo-density reduces to pseudo-density which is a constant determined by the values of pressure and entropy at top and bottom of the atmosphere considered. From (3.78), the last two entries in (5.54) imply

$$\frac{\partial \bar{M}^*}{a\partial\Phi} = 0, \quad \frac{\partial \bar{M}^*}{a\cos\Phi\partial\Lambda} = 0, \tag{5.55}$$

which indicates that $\bar{U} = \bar{V} = 0$ by referring (3.58)–(3.60). We first linearize the fundamental predictive equation (3.73) about the basic state described above. After a few steps of manipulations, we obtain

$$\frac{\partial}{\partial T}\left(\frac{\sigma^{*\prime}}{\sigma_0}\right) - \frac{1}{2\Omega a \sin^2\Phi}\frac{\partial M^{*\prime}}{a\partial\Lambda} = 0. \tag{5.56}$$

The relationship between $\sigma^{*\prime}$ and $M^{*\prime}$ can be found from the invertibility principle (3.77) or (3.82a). Before we linearize (3.77), we note that

$$\bar{\lambda} = \Lambda, \quad \overline{\cos\phi} = \cos\Phi \quad (\text{or} \quad \overline{\sin\phi} = \sin\Phi), \tag{5.57}$$

$$\lambda' = \lambda - \bar{\lambda} = -\frac{v_g'}{2\Omega a \sin\Phi\cos\Phi}, \tag{5.58}$$

$$\sin\phi' = \frac{u_g'\cos\Phi}{\Omega a(2\sin\Phi + \sin\phi')} = \frac{u_g'\cos\Phi}{2\Omega a\sin\Phi}, \tag{5.59}$$

by linearizing the formulae of geostrophic longitude and potential latitude. The following relations are also useful to note. By the definition of pseudodensity in entropy space, we have

$$\bar{\sigma}^* = \sigma_0 = -\frac{\partial \bar{p}}{\partial S}. \tag{5.60}$$

From the equation of state, pressure p is so related to the Bernoulli function by

$$\bar{p} = \bar{\rho}\mathsf{R}\bar{T} = \rho_0 \mathsf{R}\frac{\partial \bar{M}^*}{\partial S}$$

that

$$\frac{\partial \bar{p}}{\partial \sin\Phi} = \rho_0 \mathsf{R}\frac{\partial}{\partial S}\left(\frac{\partial \bar{M}^*}{\partial \sin\Phi}\right) = 0, \quad \frac{\partial \bar{p}}{\partial \Lambda} = \rho_0 \mathsf{R}\frac{\partial}{\partial S}\left(\frac{\partial \bar{M}^*}{\partial \Lambda}\right) = 0 \tag{5.61}$$

are true. With all these relations to be noted, we can now linearize the fundamental diagnostic equation (3.77):

$$\begin{aligned}
&\frac{\partial(\bar{\lambda}+\lambda', \overline{\sin\phi}+\sin\phi', \bar{p}+p')}{\partial(\Lambda, \sin\Phi, S)} + \bar{\sigma}^* + \sigma^{*\prime} \\
= \;& \frac{\partial(\sin\Phi + \sin\phi', \bar{p}+p')}{\partial(\sin\Phi, S)} + \frac{\partial(\lambda', \sin\Phi, \bar{p})}{\partial(\Lambda, \sin\Phi, S)} + \bar{\sigma}^* + \sigma^{*\prime} \\
= \;& \frac{\partial(\sin\Phi, \bar{p})}{\partial(\sin\Phi, S)} + \frac{\partial(\sin\phi', \bar{p})}{\partial(\sin\Phi, S)} + \frac{\partial(\sin\Phi, p')}{\partial(\sin\Phi, S)} + \frac{\partial(\lambda', \bar{p})}{\partial(\Lambda, S)} + \bar{\sigma}^* + \sigma^{*\prime} \\
= \;& \frac{\partial \bar{p}}{\partial S} + \frac{\partial \bar{p}}{\partial S}\frac{\partial \sin\phi'}{\partial \sin\Phi} - \frac{\partial \sin\phi'}{\partial S}\frac{\partial \bar{p}}{\partial \sin\Phi} + \frac{\partial p'}{\partial S} + \frac{\partial \lambda'}{\partial \Lambda}\frac{\partial \bar{p}}{\partial S} - \frac{\partial \lambda'}{\partial S}\frac{\partial \bar{p}}{\partial \Lambda} + \sigma_0 + \sigma^{*\prime} = 0,
\end{aligned}$$

which can be written

$$\frac{\sigma^{*\prime}}{\sigma_0} = \frac{\partial}{\partial \sin\Phi}\left(\frac{u_g' \cos\Phi}{2\Omega a \sin\Phi}\right) - \frac{1}{\sigma_0}\frac{\partial p'}{\partial S} + \frac{\partial}{\partial \Lambda}\left(-\frac{v_g'}{2\Omega a \sin\Phi \cos\Phi}\right), \tag{5.62}$$

by rearranging terms. It has been pointed out before that u_g', v_g' and p' are all related to the Bernoulli function $M^{*\prime}$ by gradient, geostrophic balanced equations, and by hydrostatic equation and equation of state, respectively. The derivations are briefly shown below. From (3.78), we can immediately write down

$$u_g' = -\frac{1}{f}\frac{\partial M^{*\prime}}{a\partial\Phi}, \tag{5.63}$$

$$v_g' = \frac{1}{f}\frac{\partial M^{*\prime}}{a\cos\Phi\partial\Lambda}, \tag{5.64}$$

since $\bar{u}_g = \bar{v}_g = 0$, and where $f = 2\Omega \sin \Phi$. From hydrostatic equation and equation of state, we have

$$\frac{\partial M^*}{\partial S} = \frac{p}{\rho \mathsf{R}} = \frac{p e^{S/c_p}}{\rho_0 \mathsf{R}} \left(\frac{p}{p_0}\right)^{1-\kappa} = T_0 \left(\frac{p}{p_0}\right)^{\kappa} e^{S/c_p}$$

so that

$$\frac{\partial M^{*\prime}}{\partial S} = \kappa \frac{p'}{\rho_0 \mathsf{R}} \tag{5.65}$$

where $\kappa = \mathsf{R}/c_p$.

Substituting (5.63)–(5.65) into (5.62), we obtain the final form of the linearized invertibility principle:

$$\frac{\sigma^{*\prime}}{\sigma_0} = \frac{\partial}{\cos\Phi \partial\Phi}\left[-\frac{\cos\Phi}{(2\Omega a \sin\Phi)^2}\frac{\partial M^{*\prime}}{\partial\Phi}\right] - \frac{\rho_0 \mathsf{R}}{\kappa\sigma_0}\frac{\partial^2 M^{*\prime}}{\partial S^2} + \frac{\partial}{\partial\Lambda}\left[\frac{1}{(2\Omega a \sin\Phi\cos\Phi)^2}\frac{\partial M^{*\prime}}{\partial\Lambda}\right], \tag{5.66}$$

which gives the one-to-one relation between perturbation potential pseudodensity $\sigma^{*\prime}$ and perturbation Bernoulli function $M^{*\prime}$.

We now combine the predictive equation (5.56) and the diagnostic equation (5.66) together to yield

$$\frac{\partial}{\partial T}\left[\frac{\partial^2 M^{*\prime}}{\cos^2\Phi\partial\Lambda^2} + \frac{\sin^2\Phi}{\cos\Phi}\frac{\partial}{\partial\Phi}\left(\frac{\cos\Phi}{\sin^2\Phi}\frac{\partial M^{*\prime}}{\partial\Phi}\right) + \frac{4\Omega^2 a^2\sin^2\Phi}{\Gamma_0\sigma_0}\frac{\partial^2 M^{*\prime}}{\partial S^2}\right] + 2\Omega\frac{\partial M^{*\prime}}{\partial\Lambda} = 0, \tag{5.67}$$

where we have denoted $\Gamma_0 = \kappa/(\mathsf{R}\rho_0) = 1/(c_p\rho_0)$.

We assume that the vertical structure of the fluid motion that we studied is separatable from the horizontal one so that the allowable solution of (5.67) takes the form

$$M^{*\prime}(\Lambda, \Phi, S, T) = \mathcal{M}(\Phi)\cos\left[\frac{\alpha_m(S_T - S)}{S_T - S_B}\right] e^{i(s\Lambda + \sigma T)}, \tag{5.68}$$

where α_m is the separation constant which will be discussed as we proceed, S_T and S_B are the values of entropy at top and bottom of model domain.

The substitution of the wave solution (5.68) leads to the meridional structure equation

$$\mathcal{L}(\mathcal{M}) = \epsilon_m \mathcal{M}, \tag{5.69}$$

where $\mathcal{L}$ denotes the linear operator

$$\mathcal{L} = \frac{d}{\cos\Phi d\Phi}\left(\frac{\cos\Phi}{\sin^2\Phi}\frac{d}{d\Phi}\right) + \frac{1}{\sin^2\Phi}\left(\frac{s}{\omega} - \frac{s^2}{\cos^2\Phi}\right), \tag{5.70}$$

where $\omega = \sigma/2\Omega$ is the normalized frequency. There should be no surprise that (5.70) is identical to the result derived by Magnusdottir and Schubert (1991) from semigeostrophic theory on the sphere, because in the limit of linearization about a basic state of rest the gradient balance reduces to geostrophic balance. Such a limit has the same asymptotic property as that in the limit of infinitely small flow curvature. Therefore the reduction of the mix-balanced theory developed in Chapter 3 to the semigeostrophic theory on the sphere can be expected in such a limit. This is not to say, however, that the mixed-balance theory is redundant to the semigeostrophic theory of Magnusdottir and Schubert (1991). In the cases of any nonresting basic state, the mixed-balance theory would be expected to produce quite different results, presumably better ones, than those by semigeostrophic theory, even in the linear context, e.g., the analysis conducted in section 5.1 of this chapter. Nevertheless, to complete the study, let us proceed with the current analysis. (5.70) should be compared with the meridional operator (5.53) in Laplace's tidal theory discussed in previous subsection. The only difference between the two is the neglect of the ω^2 factors. In the current balanced model, disappearance of this factor is closely related to the filtering of the westward and eastward propagating gravity waves. Let us go on to calculate and compare the eigenmodes from the two different models to see what the effect is on the solutions by this neglect.

In (5.69), the constant ϵ_m denotes Lamb's parameter, which is

$$\epsilon_m = \frac{4\Omega^2 a^2}{c_m^2} \tag{5.71}$$

where $c_m = c/\alpha_m$ is the phase speed of gravity waves for different vertical modes, and $c^2 = \Gamma_0\sigma_0(S_T - S_B)^2$. The separation constant α_m can be obtained from the linearized lower vortical boundary condition, which is in the form

$$c_p \frac{\partial M^{*\prime}}{\partial S} - M^{*\prime} = 0 \quad \text{at} \quad S = S_B \tag{5.72}$$

By substituting the wave-form solution (7.68) into this equation, we obtain

$$\alpha_m \tan \alpha_m = \frac{S_T - S_B}{c_p}. \tag{5.73}$$

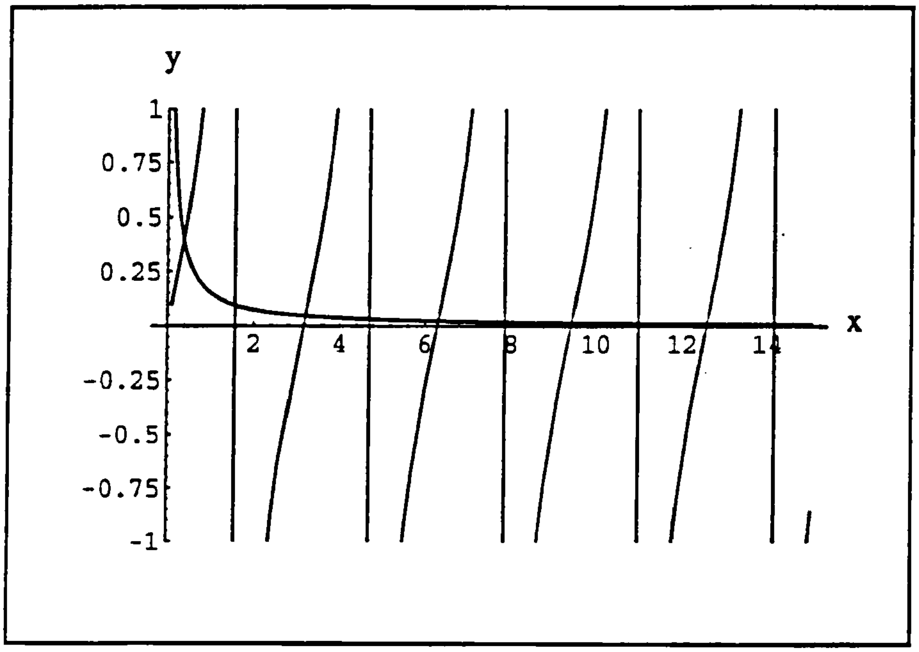


Figure 5.6: Determination of the separation constant from curves: $y = \tan x$ and $y = b/x$, where $b = (S_T - S_B)/c_p$. We choose $p_T = 22.5$ kPa, $\theta_T = 333$ K, $p_B = 100$ kPa, $\theta_B = 287$ K and $\rho_0 = 0.8$ kgm^{-3}.

	m=0	1	2	3	4
α_m	0.1198π	1.0148π	2.0075π	3.0052π	4.0037π
c_m	318.9ms^{-1}	37.6ms^{-1}	19.0ms^{-1}	12.7ms^{-1}	9.5ms^{-1}
ϵ_m	8.5	610	2382	5348	9558

Table 5.6: Computed vertical separation constants, external and internal gravity wave phase velocities and Lamb's parameters.

The solutions of this transcendental equation are the abscissa values of the intersections of two curves (shown in Figure 5.6): $y = \tan x$ and $y = b/x$, where $b = (S_T - S_B)/c_p$. In this study we choose p_T = 22.5 kPa, θ_T =333 K, p_B=100 kPa, θ_B =287 K from the U.S. Standard Atmosphere, and ρ_0 = 0.8 kgm^{-3} is taken the averaged value of density profile in troposphere. With these values, we then have b = 0.1487, c = 120 ms^{-1}. The computed α_m, phase speed c_m and Lamb's parameter ϵ_m for different vertical modes are listed in Table 5.6.

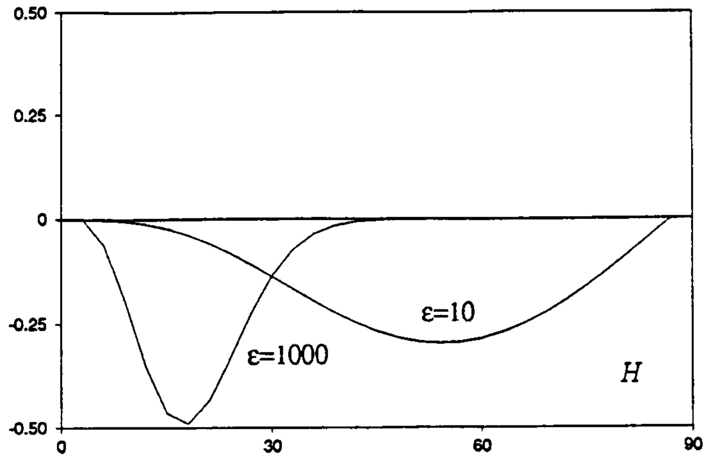
In Longuet-Higgins (1968), the eigensolutions calculated from the shallow water primitive equation model are presented in terms of constant Lamb's parameters with decreasing powers of ten. In comparison with Table 5.6, we may approximately regard his solutions for $\epsilon_0 = 10$ as the external mode, and those for $\epsilon_1 = 1000$ as the first internal mode, etc. For convenience of comparison of our results with the Laplace tidal equation results of Longuet-Higgins (1968), let us solve the meridional structure equation (5.69) for $\epsilon_0 = 10$ and $\epsilon_1 = 1000$. Equation (5.69) is first rewritten in the form of a standard eigenvalue problem, then discretized in the meridional direction on a uniform grid. The results after solving the eigenvalue problem are shown in Table 5.7, along with the corresponded modes listed by side from the primitive equation solution of Longuet-Higgins (1968).

The conclusions from this table are obvious: the balanced model developed in Chapter 3 filters out all the wave modes but those of Rossby-Haurwitz type, and such a filtered model predicts the slow motions that are fairly accurate compared with those by the primitive equation model. The balanced mass and winds fields are presented in Figures 5.7 and 5.8. In comparison with Figure 5.3, we can see that the differences of eigenvectors calculated from both models are minor.

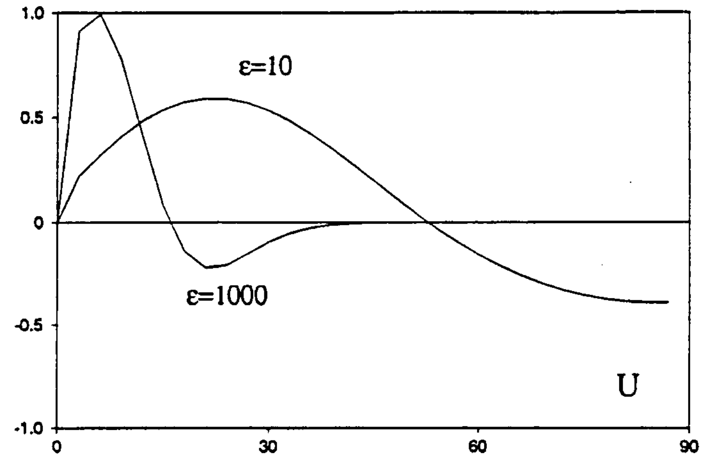
	$\epsilon_m = 10$(external mode)		$\epsilon_m = 10^3$(1st internal mode)	
	Balanced	PE	Balanced	PE
		$s = 1$		
$n - s = 2$	0.057802	0.058026	0.006379	0.006397
$n - s = 4$	0.028473	0.028377	0.003690	0.003628
		$s = 2$		
$n - s = 2$	0.082076	0.082513	0.012533	0.012542
$n - s = 4$	0.042925	0.042775	0.007286	0.007166
		$s = 3$		
$n - s = 2$	0.088976	0.089365	0.018179	0.018215
$n - s = 4$	0.049915	0.049728	0.010700	0.010529
		$s = 4$		
$n - s = 2$	0.088430	0.088704	0.023169	0.023246
$n - s = 4$	0.052870	0.052658	0.013862	0.013649
		$s = 5$		
$n - s = 2$	0.084948	0.085126	0.027408	0.027532
$n - s = 4$	0.053619	0.053390	0.016718	0.016472

Table 5.7: The eigenfrequencies of Rossby-Haurwitz waves computed from the mixed-balance model on the sphere are compared with those from Laplace's tidal equations (Longuet-Higgins, 1968).

(a)



(b)



(c)

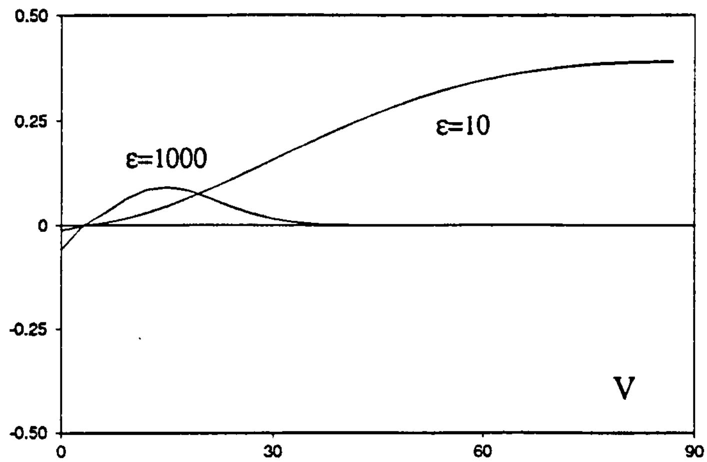


Figure 5.7: Eigenfunctions of waves of the second class from the mixed-balance model for zonal wavenumber 1.

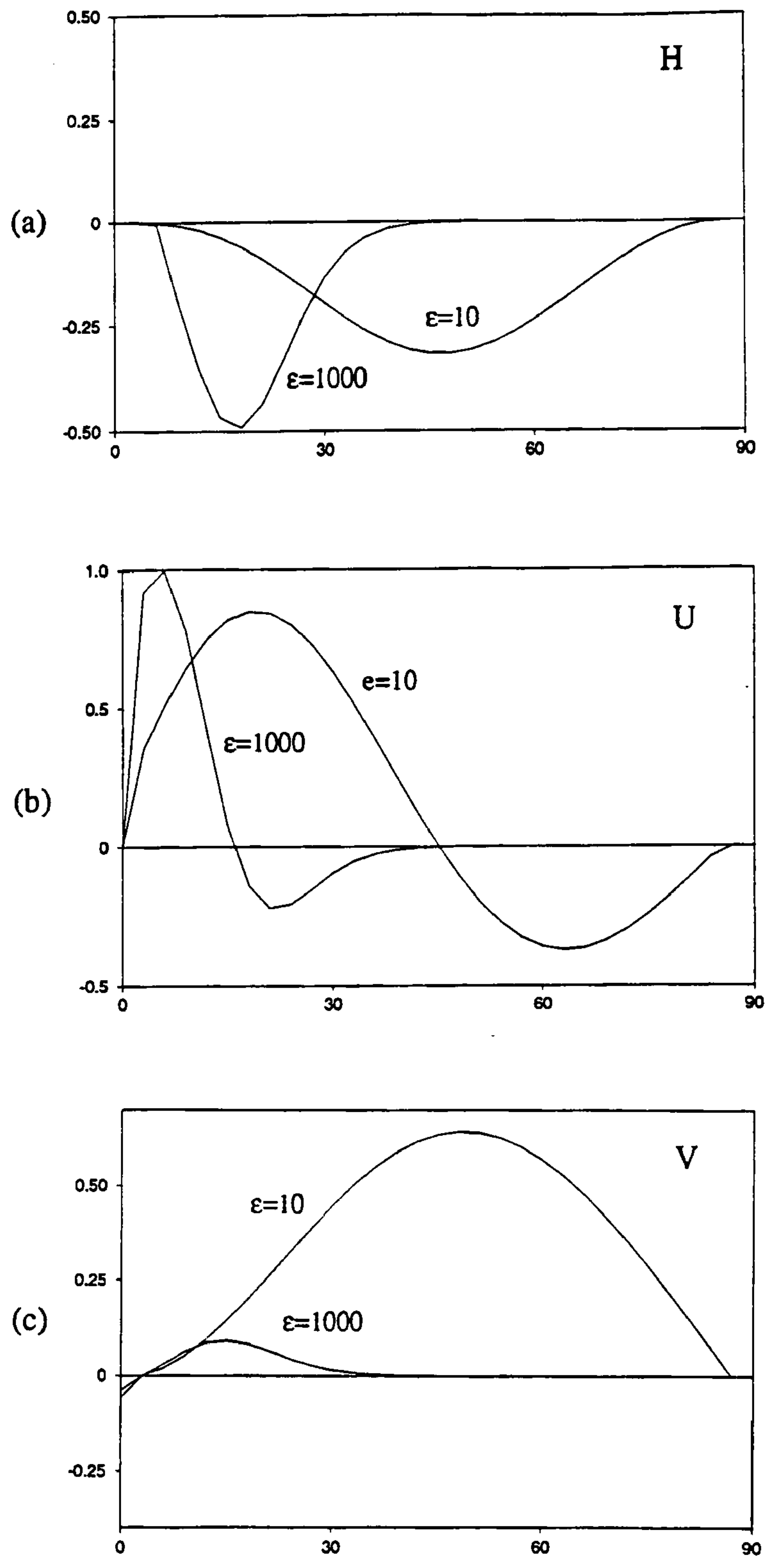


Figure 5.8: Eigenfunctions of waves of the second class from the mixed-balance model for zonal wavenumber 2.

Chapter 6

THE THEOREMS FOR COMBINED BAROTROPIC AND BAROCLINIC INSTABILITY

One of the fundamental problems in fluid dynamics is the stability of a basic flow, i.e., the transition of a laminar flow to a highly turbulent one. The development of synoptic weather systems has been conceived as a dynamical process of this kind. This problem was first studied more than a hundred years ago when Rayleigh (1880) discovered that a two dimensional incompressible flow can be unstable if the velocity profile of the flow has an inflection point. This so-called Rayleigh necessary condition is a well-known theorem to meteorologists and oceanographers, and such an instability mechanism is referred to as the barotropic instability. Kuo (1949) generalized this result to the case where the earth's rotation and geometry are taken into account. Thus, quite similar to Rayleigh's theorem, Kuo's condition can be stated as: the necessary condition for a basic flow to be unstable is that the gradient of absolute vorticity associated with this flow, $\beta - U_{yy}$, changes sign somewhere in the domain. The stability theorem for two dimensional incompressible flow under finite-amplitude perturbations was first discovered by the Russian scientist Arnol'd (1965, 1966). The idea of this generalized finite amplitude stability theorem is crucially dependent upon the fact that there exists a quantity, the so-called "Casimir", for the conservative dynamical system of the Ertel-Rossby type that the proper choice of the form of such a Casimir leads to a *priori* convexity estimates of a normed flow field. This nonlinear result deals with stability in the sense of Liapunov, i.e., the finite-amplitude disturbance at an arbitrary time t is bounded by the initial one. This theorem was generalized by McIntyre and Shepherd (1987) to a rotating fluid on the β-plane and sphere.

Charney (1948) introduced the quasi-geostrophic system in which divergence and vertical structure of the flow become permissible physics. This theory successfully treats atmospheric and oceanic motions with strong baroclinicity, and the baroclinic instability comes into the dynamical system as a new instability mechanism. In many realistic physical situations, of course, flows are not often separable into purely barotropic or purely baroclinic. Thus, in 1962 Charney and Stern derived a more general stability criterion which combines both the barotropic and baroclinic instability mechanisms. Their condition for a basic flow with both vertical and horizontal shear to be unstable is that the meridional gradient of potential vorticity of the flow either vanishes or changes sign in the domain considered. The nonlinear version of this theorem was found in Shepherd (1988, 1989), which again is a natrual extension of Arnol'd's result so that the stablity criterion is in the sense of Liapunov.

With the development of more sophisticated balanced systems, corresponding stability theorems have been investigated to include more physics. Eliassen (1983) generalized the Charney-Stern theorem for a set of balanced equations with geostrophic momentum approximations (GM). The Charney-Stern theorems associated with the semigeostrophic theory on the β-plane and sphere were derived by Magnusdottir and Schubert (1990, 1991). The approach used in these recent stability analyses is different from that of Charney and Stern (1962) in that the fixed structure of the normal modes is not assumed in order to derive such a stability theorem. Instead, the Lagrangian concept of particle displacements about the mean flow, originally used by Taylor (1915), is adopted. These theorems have the same stability arguments as those of Charney and Stern's except for the fact that the potential vorticity in Charney and Stern (1962) is the quasi-geostrophic one, while the potential vorticity in the stability theorem derived by Eliassen is more of the Ertel's type and Magnusdottir and Schubert expressed their theorem in terms of reciprocal of Ertel potential vorticity. This is simply due to the fact that GM and SG include more physics than QG theory.

So far we have just discussed the stability theorems associated with balanced dynamics. The disturbances, therefore, are confined to the geostrophic or Rossby wave types.

What stability argument can we make if the dynamical system is general enough to contain not only geostrophic disturbances but also disturbances of inertia-gravity type? Such a dynamical system is, of course, used in meteorological practice most commonly nowadays. Ripa (1983) derived the stability conditions for a shallow water primitive equation system on the β-plane and on the sphere, and later he generalized his stability conditions to a multi-layer model (Ripa, 1991). His stability condition understandably comprises two parts: (1) the products of velocity profile and the meridional gradient of potential vorticity is always a nonpositive value; (2) the flow is always in the "subsonic" condition (the Froude number $F \leq 1$). The first part of this condition is quite similar to the Rayleigh-Fjørtoft argument, thus being conjectured as the condition imposed on Rossby wave disturbances. The second part essentially requires a high phase speed of gravity wave propagation, which seems to be related to the condition on the nongeostrophic disturbances. A historical summary of stability theories and their present state of development are given in Table 6.1.

A noteworthy connection is found between the stability analyses discussed above and the theoretical framework of generalized Eliassen-Palm theorems by Andrews and McIntyre (1976, 1978), Andrews (1983), Haynes (1989). The latter category of studies showed that for forced or unforced, for linear or nonlinear, for the quasi-geostrophic or the primitive equation systems on the f-plane or on the β-plane or on the sphere, one can always derive a generalized wave-activity relation in the form:

$$\frac{\partial A}{\partial t} + \nabla \cdot \mathbf{F} = S, \tag{6.1}$$

where A is the wave-activity, $\mathbf{F}$ is the Eliassen-Palm flux, and S is the sources and sinks of A. For small-amplitude (or even in certain finite-amplitude circumstances), conservative waves, S is effectively zero. The conservation relation then obtained naturally leads to the local stability arguments, because in the small-amplitude limit the wave-activity is expressed as $A \simeq \frac{1}{2}\eta'^2/Q_y$, where η' is particle displacement, Q_y is the meridional gradient of vorticity (or potential vorticity) of mean flow. If there is no net transport of E-P flux across the region considered, then wave-activity is a locally conserved quantity. Thus, for a growing disturbance the meridional gradient of vorticity (or PV) must change sign, which is exactly the stability statement made in the Rayleigh and Charney-Stern theorems.

Dynamics	Linear		Nonlinear	
Nondivergent	Rayleigh (2-D incompressible)	1880	Arnol'd	1966
	Taylor (2-D incompressible)	1915	(2-D incompressible)	
	Fjørtoft	1950	McIntyre & Shepherd	1987
	Kuo (β-plane, sphere)	1949	(β-plane, sphere)	
Quasi-geostrophic	Charney & Stern	1962	Shepherd	1988
	(β-plane, sphere)		(two-layers)	
			Shepherd	1989
			(continuously stratified)	
Semi-geostrophic	Eliassen (f-plane)	1983	?	
	Magnusdottir & Schubert	1990		
	(β-plane)			
	Magnusdottir & Schubert	1991		
	(sphere)			
Mixed-balance	present work		?	
	(f-plane)			
	present work			
	(sphere)			
Primitive equation	Ripa (shallow water)	1983	?	
	Ripa	1991		
	(multiple finite layers)			

Table 6.1: Summary of linear and nonlinear stability theory for various types of dynamical systems. Question marks imply no known theory exists.

Since our new balanced systems developed in chapter 2 and 3 generalize the axisymmetric and zonally symmetric theories, the eddy motions come into existence in the dynamical systems. An important and obvious problem is to determine in what situations these eddies can amplify, or in what situations they cannot. Furthermore, the theories we proposed are higher order balanced systems than QG and SG in the sense that the curvature effect is captured in the dynamics. Therefore, the generalization of stability theory for growing disturbances superimposed on highly curved mean flows, the circular flow for instance, is by no means physically trivial. Under these considerations, we derive stability theorems of the Charney-Stern type for mixed-balance systems on an f-plane (section 6.1) and on a sphere (section 6.2). The position of the current work in relation to other theoretical stability analyses is shown in Table 6.1.

6.1 The generalized wave-activity relation and the Charney-Stern theorem with the 3-D balanced vortex theory

The primary governing equations for this section are the predictive equation (2.76) and the diagnostic equation (2.80) derived in chapter 2. These two equations coupled with their boundary conditions (see Table 2.1) form the basic theoretical framework of the mixed-balance system on an f-plane.

Let us first consider a basic state of axisymmetry (quantities denoted by overbars) and a small perturbation (quantities denoted by primes) about such a basic state. We also neglect frictional and diabatic effects. A linearized potential pseudodensity equation can be derived from (2.76), which gives

$$\frac{\partial \sigma^{*\prime}}{\partial T} + \frac{1}{f}\frac{\partial \bar{M}^*}{\partial R}\frac{\partial \sigma^{*\prime}}{R\partial \Phi} - \frac{1}{f}\frac{\partial M^{*\prime}}{R\partial \Phi}\frac{\partial \bar{\sigma}^*}{\partial R} = 0. \tag{6.2}$$

Equation (6.2) may also be written in the form

$$\frac{\mathcal{D}\sigma^{*\prime}}{\mathcal{D}t} + u_g'\frac{\partial \bar{\sigma}^*}{\partial R} = 0, \tag{6.3}$$

by introducing the notations $\mathcal{D}/\mathcal{D}t = \partial/\partial T + \bar{V}\partial/R\partial\Phi$ and $fu_g' = -\partial M^{*\prime}/R\partial\Phi$. $\bar{V}$ is the transformed tangential basic state wind whose definition is given by (2.63) and (2.64).

In analogy to Eliassen (1983), let us define the radial geostrophic particle displacement η' such that

$$u_g' = \frac{\mathcal{D}\eta'}{\mathcal{D}t}. \tag{6.4}$$

Since $\bar{\sigma}^*$ is independent of $\mathcal{T}$ and Φ, using this equation we can integrate (6.3) to obtain

$$\sigma^{*\prime} + \eta' \frac{\partial \bar{\sigma}^*}{\partial R} = 0. \tag{6.5}$$

Multiplying (6.5) by u_g' and taking the azimuthal average, we obtain

$$\frac{\partial}{\partial \mathcal{T}} \left(\tfrac{1}{2} \overline{\eta'^2} \frac{\partial \bar{\sigma}^*}{\partial R} \right) + \overline{u_g' \sigma^{*\prime}} = 0. \tag{6.6}$$

The quantity inside the time derivative of the first term is the linearized wave-activity written in terms of potential pseudodensity. The physics depicted by this equation is clear: the local time rate of change of wave activity in a circular vortex is determined by the net radial transport of potential pseudodensity. We can further change the second term in (6.6) into a flux-divergent form, thus expressing (6.6) in the standard wave-activity relation of the form (6.1). In order to do this, let us linearize the invertibility principle (2.80). We first note that the basic state is axisymmetric, $\bar{u}_g = 0$, and therefore $\bar{\phi} = \Phi$. Eq. (2.80) for the basic state can then be written

$$\bar{\sigma}^* = -\frac{\partial(\frac{1}{2}\bar{r}^2, \bar{p})}{\partial(\frac{1}{2}R^2, S)}. \tag{6.7}$$

Linearizing (2.80) and using (6.7), we have

$$\begin{aligned}
&\frac{\partial(\frac{1}{2}\bar{r}^2 + \frac{1}{2}r^{2\prime}, \bar{\phi} + \phi', \bar{p} + p')}{\partial(\frac{1}{2}R^2, \Phi, S)} + \bar{\sigma}^* + \sigma^{*\prime} \\
= \; & \frac{\partial(\frac{1}{2}\bar{r}^2 + \frac{1}{2}r^{2\prime}, \Phi, \bar{p} + p')}{\partial(\frac{1}{2}R^2, \Phi, S)} + \frac{\partial(\frac{1}{2}\bar{r}^2, \phi', \bar{p})}{\partial(\frac{1}{2}R^2, \Phi, S)} + \bar{\sigma}^* + \sigma^{*\prime} \\
= \; & \frac{\partial(\frac{1}{2}\bar{r}^2, p')}{\partial(\frac{1}{2}R^2, S)} + \frac{\partial(\frac{1}{2}r^{2\prime}, \bar{p})}{\partial(\frac{1}{2}R^2, S)} + \frac{\partial \phi'}{\partial \Phi} \frac{\partial(\frac{1}{2}\bar{r}^2, \bar{p})}{\partial(\frac{1}{2}R^2, S)} + \sigma^{*\prime} = 0.
\end{aligned}$$

Multiplying by fRu_g' and then taking the azimuthal average, the third term drops because

$$\phi' = \phi - \bar{\phi} = \frac{u_g'}{fR} \tag{6.8}$$

from the geostrophic azimuth relation (2.56). Rearranging the rest of the terms, we have

$$
\begin{aligned}
\overline{fRu'_g\sigma^{*\prime}} &= \overline{fRu'_g\left(\frac{\partial \bar{p}}{R\partial R}\frac{\partial(\frac{1}{2}r^{2\prime})}{\partial S} - \frac{\partial \bar{p}}{\partial S}\frac{\partial(\frac{1}{2}r^{2\prime})}{R\partial R}\right)} + \overline{fRu'_g\left(\frac{\partial p'}{R\partial R}\frac{\partial(\frac{1}{2}\bar{r}^2)}{\partial S} - \frac{\partial p'}{\partial S}\frac{\partial(\frac{1}{2}\bar{r}^2)}{R\partial R}\right)} \\
&= \frac{\partial}{R\partial R}\left(\frac{\partial(\frac{1}{2}\bar{r}^2)}{\partial S}\overline{fRu'_gp'} - \frac{\partial \bar{p}}{\partial S}\overline{fRu'_g(\tfrac{1}{2}r^{2\prime})}\right) \\
&\quad + \frac{\partial}{\partial S}\left(\frac{\partial \bar{p}}{R\partial R}\overline{fRu'_g(\tfrac{1}{2}r^{2\prime})} - \frac{\partial(\frac{1}{2}\bar{r}^2)}{R\partial R}\overline{fRu'_gp'}\right) \\
&\quad + \overline{(\tfrac{1}{2}r^{2\prime})\frac{\partial}{R\partial R}\left(\frac{\partial \bar{p}}{\partial S}fRu'_g\right)} - \overline{p'\frac{\partial}{R\partial R}\left(\frac{\partial(\frac{1}{2}\bar{r}^2)}{\partial S}fRu'_g\right)} \\
&\quad + \overline{p'\frac{\partial}{\partial S}\left(\frac{\partial(\frac{1}{2}\bar{r}^2)}{R\partial R}fRu'_g\right)} - \overline{(\tfrac{1}{2}r^{2\prime})\frac{\partial}{\partial S}\left(\frac{\partial \bar{p}}{R\partial R}fRu'_g\right)}
\end{aligned}
$$

We define the last four terms in this expression as residual $\mathcal{R}$, and calculate this residual explicitly as follows:

$$
\begin{aligned}
\mathcal{R} &= \overline{\tfrac{1}{2}r^{2\prime}\left[\frac{\partial}{R\partial R}\left(\frac{\partial \bar{p}}{\partial S}fRu'_g\right) - \frac{\partial}{\partial S}\left(\frac{\partial \bar{p}}{R\partial R}fRu'_g\right)\right]} \\
&\quad \overline{+p'\left[\frac{\partial}{\partial S}\left(\frac{\partial(\frac{1}{2}\bar{r}^2)}{R\partial R}fRu'_g\right) - \frac{\partial}{R\partial R}\left(\frac{\partial(\frac{1}{2}\bar{r}^2)}{\partial S}fRu'_g\right)\right]} \\
&= \overline{\tfrac{1}{2}r^{2\prime}\left[\frac{\partial \bar{p}}{\partial S}\frac{\partial}{R\partial R}(fRu'_g) - \frac{\partial \bar{p}}{R\partial R}\frac{\partial}{\partial S}(fRu'_g)\right]} \\
&\quad \overline{+p'\left[\frac{\partial(\frac{1}{2}\bar{r}^2)}{R\partial R}\frac{\partial}{\partial S}(fRu'_g) - \frac{\partial(\frac{1}{2}\bar{r}^2)}{\partial S}\frac{\partial}{R\partial R}(fRu'_g)\right]} \\
&= \overline{\tfrac{1}{2}r^{2\prime}\left[\frac{\partial \bar{p}}{\partial S}\frac{\partial}{\partial \Phi}\left(\frac{\partial M^{*\prime}}{R\partial R}\right) - \frac{\partial \bar{p}}{R\partial R}\frac{\partial}{\partial \Phi}\left(\frac{\partial M^{*\prime}}{\partial S}\right)\right]} \\
&\quad \overline{+p'\left[\frac{\partial(\frac{1}{2}\bar{r}^2)}{R\partial R}\frac{\partial}{\partial \Phi}\left(\frac{\partial M^{*\prime}}{\partial S}\right) - \frac{\partial(\frac{1}{2}\bar{r}^2)}{\partial S}\frac{\partial}{\partial \Phi}\left(\frac{\partial M^{*\prime}}{R\partial R}\right)\right]}
\end{aligned}
$$

where the last two lines have been obtained by using the perturbed geostrophic wind relation (2.82), and switching the order of differentiation. It is not difficult to obtain the mean and perturbed hydrostatic equations through the standard linearization procedure, which gives

$$\frac{\partial \bar{M}^*}{\partial S} = \frac{\bar{p}}{\mathsf{R}\bar{\rho}}, \tag{6.9}$$

$$\frac{\partial M^{*\prime}}{\partial S} = \kappa\frac{p'}{\mathsf{R}\bar{\rho}}. \tag{6.10}$$

where R is the gas constant, and $\kappa = \mathsf{R}/c_p$. From the potential radius formula (2.55), we can get

$$\tfrac{1}{2}\bar{r}^2 = \tfrac{1}{2}R^2 - \frac{\bar{r}\bar{v}_g}{f}, \tag{6.11}$$

and

$$\tfrac{1}{2}r^{2\prime} = -\frac{(rv_g)'}{f}. \tag{6.12}$$

Linearizing the gradient wind relations, the first entry of (2.82), and using (6.11)–(6.12), we obtain the mean and perturbed gradient wind in the forms

$$f\bar{r}\bar{v}_g = \frac{\bar{r}^2}{R}\frac{\partial \bar{M}^*}{\partial R}, \tag{6.13}$$

$$\frac{\partial M^{*\prime}}{R\partial R} = -\frac{f^2R^2}{2\bar{r}^4}r^{2\prime}. \tag{6.14}$$

On substituting (6.10) and (6.14) into the residual expression, the first and the third terms are dropped, yielding

$$\mathcal{R} = \frac{\kappa}{\mathsf{R}\bar{p}}\frac{\partial \bar{p}}{R\partial R}\overline{(\tfrac{1}{2}r^{2\prime})\frac{\partial p'}{\partial \Phi}} - \frac{f^2R^2}{\bar{r}^4}\frac{\partial(\frac{1}{2}\bar{r}^2)}{\partial S}\overline{p'\frac{\partial(\frac{1}{2}r^{2\prime})}{\partial \Phi}}. \tag{6.15}$$

Further cancellation in this equation is noted by deriving a mean thermal wind relation. Let us take the radial derivative of the mean hydrostatic equation (6.9), take the vertical derivative of the mean gradient wind equation (6.13), and combine the resultant equations to get

$$\frac{\kappa}{\mathsf{R}\bar{p}}\frac{\partial \bar{p}}{R\partial R} + \frac{f^2R^2}{\bar{r}^4}\frac{\partial(\frac{1}{2}\bar{r}^2)}{\partial S} = 0. \tag{6.16}$$

Substituting (6.16) into (6.15), we finally obtain

$$\mathcal{R} = -\frac{\kappa}{\mathsf{R}\bar{p}}\frac{\partial \bar{p}}{R\partial R}\overline{p'\frac{\partial(\frac{1}{2}r^{2\prime})}{\partial \Phi}} - \frac{f^2R^2}{\bar{r}^4}\frac{\partial(\frac{1}{2}\bar{r}^2)}{\partial S}\overline{p'\frac{\partial(\frac{1}{2}r^{2\prime})}{\partial \Phi}} = 0. \tag{6.17}$$

With this result, the linearized invertibility principle can be written into a extremely simple form:

$$fR\overline{u_g'\sigma^{*\prime}} = \nabla\cdot\mathbf{F} \tag{6.18}$$

where $\nabla = (\partial/\partial R,\ \partial/\partial S)$ is the two dimensional del operator, $\mathbf{F}$ is the combined geostrophic-and gradient E-P flux on the radial-height plane, taking the form

$$\mathbf{F} = \left(-\frac{\partial(\bar{r}\bar{v}_g)}{\partial S}\overline{u_g'p'} + \frac{\partial \bar{p}}{\partial S}\overline{u_g'(rv_g)'},\ -\frac{\partial \bar{p}}{\partial R}\overline{u_g'(rv_g)'} - [fR - \frac{\partial(\bar{r}\bar{v}_g)}{\partial R}]\overline{u_g'p'}\right). \tag{6.19}$$

On substituting (6.18) into the linearized potential pseudodensity equation (6.6), we obtain the generalized wave-activity relation for the mixed-balance system on an f-plane developed in chapter 2:

$$\frac{\partial}{\partial T}\left(\tfrac{1}{2}\overline{\eta'^2} fR\frac{\partial \bar{\sigma}^*}{\partial R}\right) + \nabla\cdot\mathbf{F} = 0, \tag{6.20}$$

For steady flow, (6.20) reduces to the Eliassen-Palm theorem. Conceptually, this equation can be used to diagnose the divergence of E-P flux through the local time rate of change of wave-activity, and therefore to study wave-mean interaction problem in balanced circular vortices. In practice, however, the Lagrangian particle displacement is difficult to calculate. Using the momentum-Casimir or energy-Casimir approach, McIntyre and Shepherd (1987) were able to generalize the analogous problem for two dimensional incompressible flow to a finite-amplitude disturbance case with wave-activity density expressed in terms of Eulerian quantities. They argued that for unifunctional basic flows, this Eulerian form of wave-activity density is easily evaluated; for multifunctional basic flows, however, the Lagrangian information is still needed to determine such a wave-activity. The generalizations of this theory to the semigeostrophic and the mixed-balance systems have not been completed at this time.

In order to obtain a stability theorem, we shall integrate (6.20) over the whole model domain. Let us first consider the vertical boundary conditions. At the top boundary, fluxes vanish since it is an isobaric surface, so that

$$p' = 0, \quad \frac{\partial \bar{p}}{\partial R} = 0, \qquad \text{at} \quad S = S_T. \tag{6.21}$$

For lower boundary, let us consider (2.86b):

$$c_p\bar{T} - \bar{M}^* + \tfrac{1}{2}\bar{v}_g^2 = 0, \qquad \text{at} \quad S = S_B \tag{6.22}$$

which, after using the equation of state and the definition of entropy in the first term and taking a radial derivative of the resultant equation, can be written

$$\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial R} - \frac{\partial \bar{M}^*}{\partial R} + \bar{v}_g\frac{\partial \bar{v}_g}{\partial R} = 0 \qquad \text{at} \quad S = S_B. \tag{6.23}$$

Note that the radial derivative of the potential radius formula yields $\partial \bar{v}_g/\partial R = fR/\bar{r} - \bar{\omega}\partial \bar{r}/\partial R$, where $\bar{\omega} = f + \bar{v}_g/\bar{r}$. Substituting this relation in (6.23) and using gradient wind equation, we can prove that

$$\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial R} - \frac{\bar{\omega}\bar{v}_g}{f\bar{r}}\frac{\partial(\frac{1}{2}f\bar{r}^2)}{\partial R} = 0 \qquad \text{at} \quad S = S_B. \tag{6.24}$$

Next we consider the perturbation part of the lower boundary condition. Multiplying (2.86b) by r^2 and linearizing it, we obtain

$$(c_p\bar{T} - \bar{M}^*)r'^2 + c_p\bar{r}^2T' - \bar{r}^2M^{*\prime} + \bar{r}\bar{v}_g(rv_g)' = 0 \quad \text{at} \quad S = S_B.$$

Multiplying this equation by u'_g and then taking an azimuthal average, we may obtain

$$-\bar{v}_g^2\overline{u'_g\tfrac{1}{2}r'^2} + \frac{\bar{r}^2}{\bar{\rho}}\overline{u'_gp'} - \bar{r}^2\overline{M^{*\prime}\frac{\partial M^{*\prime}}{R\partial\Phi}} + \bar{r}\bar{v}_g\overline{u'_g(rv_g)'} = 0 \quad \text{at} \quad S = S_B, \tag{6.25}$$

in which we have also used the perturbed hydrostatic equation (6.10) and mean lower boundary condition (6.22). The third term in (6.25) vanishes after averaging. Noting the perturbed potential radius relation (6.12), we can rewrite (6.25) as

$$\frac{1}{\bar{\rho}}\overline{u'_gp'} + \frac{\bar{\omega}\bar{v}_g}{f\bar{r}}\overline{u'_g(rv_g)'} = 0 \qquad \text{at} \quad S = S_B. \tag{6.26}$$

Now combining (6.24) and (6.26), we have proved that the E-P flux vanishes at the lower boundary, i.e.,

$$\frac{\partial \bar{p}}{\partial R}\overline{u'_g(rv_g)'} + \frac{\partial(\frac{1}{2}f\bar{r}^2)}{\partial R}\overline{u'_gp'} = 0 \qquad \text{at} \quad S = S_B. \tag{6.27}$$

For lateral boundary conditions, at vortex center we require

$$u'_g = 0 \qquad \text{at} \quad R = 0, \tag{6.28}$$

and for outer boundary condition, we choose R_L far enough so that the perturbation radial wind also vanishes, i.e.,

$$u'_g = 0 \qquad \text{at} \quad R = R_L. \tag{6.29}$$

With all these boundary conditions, namely (6.21) and (6.27)–(6.29), we now can integrate (6.20) over the radial-height plane, which yields

$$\frac{\partial}{\partial T}\iint \tfrac{1}{2}\overline{\eta'^2}fR\frac{\partial \bar{\sigma}^*}{\partial R}dRdS = 0. \tag{6.30}$$

This, of course, leads to the stability argument as follows: since the integral in (6.30) must be constant in time, in order for disturbances to grow in time, i.e., for $\overline{\eta'^2}$ to grow in time, the radial gradient of potential pseudodensity at basic state, $\partial\bar{\sigma}^*/\partial R$, must have both signs. Such a dynamical statement can be considered as a generalization of the Charney-Stern theorem for a three dimensional, balanced vortex motion. Note that in (6.30), f is a constant and the independent variable R varies in a non-negative domain. Thus, the appearance of these two quantities within the integral will not change the general stability argument above. However, such an appearance, specially for the R factor, does suggest that our mixed-balance system skews the unstable modes to the far side of vortex center. In other words, if the sign of the potential vorticity (or potential pseudodensity more precisely) were changed both near the vortex center and at large radius, the mixed-balance model would more easily pick up the unstable mode associated with the PV gradient at large radius than that associated with PV gradient near the vortex center.

6.2 The generalized wave-activity relation and the Charney-Stern theorem with the spherical mixed-balance theory

In section 6.1, we derived a generalized Eliassen-Palm theorem and a stability theorem of the Charney-Stern type for the mixed-balance system on an f-plane. In this section, we further generalize these results to full spherical geometry, i.e., we derive the Charney-Stern theorem for the mixed-balance system developed in Chapter 3.

We begin with the mixed-balance system (3.73) and (3.77) (or T3.1 and T3.2 listed in Table 3.1). For adiabatic flow, the last term in the potential pseudodensity equation (3.73) is dropped. We now consider a steady, zonally symmetric basic state with a vertically and horizontally sheared flow so that

$$\bar{u}_g = \bar{u}_g(\Phi, S), \quad \bar{v}_g = 0. \tag{6.31}$$

It is straightforward to linearize (3.73) about this basic state to obtain

$$\frac{\partial\sigma^{*\prime}}{\partial T} - \frac{\partial\bar{M}^*}{a\partial\Phi}\frac{\partial}{a\cos\Phi\partial\Lambda}\left(\frac{\sigma^{*\prime}}{2\Omega\sin\Phi}\right) + \frac{\partial M^{*\prime}}{a\partial\Lambda}\frac{\partial}{a\cos\Phi\partial\Phi}\left(\frac{\bar{\sigma}^*}{2\Omega\sin\Phi}\right) = 0. \tag{6.32}$$

We can also write (6.32) in the compact form

$$\frac{\mathcal{D}\sigma^{*\prime}}{\mathcal{D}t} + f v_g' \frac{\partial}{a\partial\Phi}\left(\frac{\bar{\sigma}^*}{f}\right) = 0, \tag{6.33}$$

where $\mathcal{D}/\mathcal{D}t = \partial/\partial T + \bar{U}\partial/a\cos\Phi\partial\Lambda$, $fv_g' = \partial M^{*\prime}/(a\cos\Phi\partial\Lambda)$ and $f = 2\Omega\sin\Phi$. $\bar{U}$ is the transformed basic state zonal wind whose definition is given in (3.58) and (3.60).

Let us now define a particle displacement associated with the meridional geostrophic velocity as

$$v_g' = \frac{\mathcal{D}\eta'}{\mathcal{D}t}. \tag{6.34}$$

Substituting this definition into (6.33) and integrating the resultant equation over time and longitude, we have

$$\sigma^{*\prime} + \eta' f \frac{\partial}{a\partial\Phi}\left(\frac{\bar{\sigma}^*}{f}\right) = 0. \tag{6.35}$$

Multiplying this equation by v_g' and then taking the zonal average, we obtain

$$\frac{\partial}{\partial T}\left[\tfrac{1}{2}\overline{\eta'^2} f \frac{\partial}{a\partial\Phi}\left(\frac{\bar{\sigma}^*}{f}\right)\right] + \overline{v_g'\sigma^{*\prime}} = 0. \tag{6.36}$$

Again, the quantity inside the time derivative is the linearized wave-activity written in terms of the inverse of potential vorticity, i.e., the potential pseudodensity normalized by the coriolis parameter. The local time rate of change of this wave-activity is clearly related to eddy transport of potential pseudodensity by meridional geostrophic disturbances. This transport can be represented as the divergence of a flux so that in a global view it has no effect on wave-activity. In order to see this, let us linearize the invertibility principle (3.77). We first note that the zonally symmetric basic state gives rise to $\bar{v}_g = 0$, and therefore $\bar{\lambda} = \Lambda$ by referring to the geostrophic longitude relation (3.51). The basic state version of (3.77) can then be written

$$\bar{\sigma}^* = -\frac{\partial(\overline{\sin\phi}, \bar{p})}{\partial(\sin\Phi, S)}. \tag{6.37}$$

Linearizing (3.77) and using (6.37), we have

$$\begin{aligned} &\frac{\partial(\bar{\lambda}+\lambda', \overline{\sin\phi}+\sin\phi', \bar{p}+p')}{\partial(\Lambda, \sin\Phi, S)} + \bar{\sigma}^* + \sigma^{*\prime} \\ =\ & \frac{\partial(\Lambda, \sin\Phi + \sin\phi', \bar{p}+p')}{\partial(\Lambda, \sin\Phi, S)} + \frac{\partial(\lambda', \overline{\sin\phi}, \bar{p})}{\partial(\Lambda, \sin\Phi, S)} + \bar{\sigma}^* + \sigma^{*\prime} \\ =\ & \frac{\partial(\sin\phi', \bar{p})}{\partial(\sin\Phi, S)} + \frac{\partial(\overline{\sin\phi}, p')}{\partial(\sin\Phi, S)} + \frac{\partial\lambda'}{\partial\Lambda}\frac{\partial(\overline{\sin\phi}, \bar{p})}{\partial(\sin\Phi, S)} + \sigma^{*\prime} = 0. \end{aligned}$$

Multiplying by $f \cos\Phi v_g'$ and then taking a zonal average, the third term disappears because

$$\lambda' = \lambda - \bar{\lambda} = -\frac{v_g'}{f a \cos\Phi} \tag{6.38}$$

from the geostrophic longitude coordinate relation (3.51). Rearranging the remaining terms and denoting $f^* = f\cos\Phi$, we have

$$\begin{aligned}
f^*\overline{v_g'\sigma^{*\prime}} &= \overline{f^* v_g'\left(\frac{\partial\bar{p}}{\partial\sin\Phi}\frac{\partial\sin\phi'}{\partial S} - \frac{\partial\bar{p}}{\partial S}\frac{\partial\sin\phi'}{\partial\sin\Phi}\right)} \\
&\quad \overline{+f^* v_g'\left(\frac{\partial\overline{\sin\phi}}{\partial S}\frac{\partial p'}{\partial\sin\Phi} - \frac{\partial\overline{\sin\phi}}{\partial\sin\Phi}\frac{\partial p'}{\partial S}\right)} \\
&= \frac{\partial}{\partial S}\left(\frac{\partial\bar{p}}{\partial\sin\Phi}\overline{f^* v_g'\sin\phi'} - \frac{\partial\overline{\sin\phi}}{\partial\sin\Phi}\overline{f^* v_g' p'}\right) \\
&\quad + \frac{\partial}{\partial\sin\Phi}\left(\frac{\partial\overline{\sin\phi}}{\partial S}\overline{f^* v_g' p'} - \frac{\partial\bar{p}}{\partial S}\overline{f^* v_g'\sin\phi'}\right) \\
&\quad \overline{+\sin\phi'\frac{\partial}{\partial\sin\Phi}\left(f^* v_g'\frac{\partial\bar{p}}{\partial S}\right)} - \overline{p'\frac{\partial}{\partial\sin\Phi}\left(f^* v_g'\frac{\partial\overline{\sin\phi}}{\partial S}\right)} \\
&\quad \overline{-p'\frac{\partial}{\partial S}\left(f^* v_g'\frac{\partial\overline{\sin\phi}}{\partial\sin\Phi}\right)} - \overline{\sin\phi'\frac{\partial}{\partial S}\left(f^* v_g'\frac{\partial\bar{p}}{\partial\sin\Phi}\right)}.
\end{aligned}$$

We define the last four terms in this expression as the residual $\mathcal{R}$, and calculate this residual explicitly as follows:

$$\begin{aligned}
\mathcal{R} &= \overline{-\sin\phi'\left[\frac{\partial}{\partial S}\left(\frac{\partial\bar{p}}{\partial\sin\Phi}f^* v_g'\right) - \frac{\partial}{\partial\sin\Phi}\left(\frac{\partial\bar{p}}{\partial S}f^* v_g'\right)\right]} \\
&\quad \overline{-p'\left[\frac{\partial}{\partial\sin\Phi}\left(\frac{\partial\overline{\sin\phi}}{\partial S}f^* v_g'\right) - \frac{\partial}{\partial S}\left(\frac{\partial\overline{\sin\phi}}{\partial\sin\Phi}f^* v_g'\right)\right]} \\
&= \overline{-\sin\phi'\left[\frac{\partial\bar{p}}{\partial\sin\Phi}\frac{\partial}{\partial S}(f^* v_g') - \frac{\partial\bar{p}}{\partial S}\frac{\partial}{\partial\sin\Phi}(f^* v_g')\right]} \\
&\quad \overline{-p'\left[\frac{\partial\overline{\sin\phi}}{\partial S}\frac{\partial}{\partial\sin\Phi}(f^* v_g') - \frac{\partial\overline{\sin\phi}}{\partial\sin\Phi}\frac{\partial}{\partial S}(f^* v_g')\right]} \\
&= \overline{-\sin\phi'\left[\frac{\partial\bar{p}}{\partial\sin\Phi}\frac{\partial}{a\partial\Lambda}\left(\frac{\partial M^{*\prime}}{\partial S}\right) - \frac{\partial\bar{p}}{\partial S}\frac{\partial}{a\partial\Lambda}\left(\frac{\partial M^{*\prime}}{\partial\sin\Phi}\right)\right]} \\
&\quad \overline{-p'\left[\frac{\partial\overline{\sin\phi}}{\partial S}\frac{\partial}{a\partial\Lambda}\left(\frac{\partial M^{*\prime}}{\partial\sin\Phi}\right) - \frac{\partial\overline{\sin\phi}}{\partial\sin\Phi}\frac{\partial}{a\partial\Lambda}\left(\frac{\partial M^{*\prime}}{\partial S}\right)\right]},
\end{aligned}$$

where the last two lines have been obtained by using the perturbed geostrophic wind relation from (3.78), and switching the derivative orders.

Similar to the previous section, we can derive the mean and perturbed hydrostatic equations through the standard linearization procedure, which gives

$$\frac{\partial \bar{M}^*}{\partial S} = \frac{\bar{p}}{\mathsf{R}\bar{\rho}}, \tag{6.39}$$

$$\frac{\partial M^{*\prime}}{\partial S} = \kappa \frac{p'}{\mathsf{R}\bar{\rho}}, \tag{6.40}$$

where R is the gas constant, and $\kappa = \mathsf{R}/c_p$.

From the potential latitude formula (3.49), we obtain

$$\overline{\cos^2 \phi} = \cos^2 \Phi - \frac{\overline{u_g \cos \phi}}{\Omega a}, \tag{6.41}$$

and

$$(\cos^2 \phi)' = -\frac{(u_g \cos \phi)'}{\Omega a}. \tag{6.42}$$

With these relations, we can now linearize the gradient wind formula, (3.36) or the second entry of (3.78) without any difficulty. The results give the mean and perturbed gradient wind relations as

$$f\overline{u_g \cos \phi} = -\frac{\overline{\cos^2 \phi}}{\cos \Phi} \frac{\partial \bar{M}^*}{a \partial \Phi}, \tag{6.43}$$

$$\frac{\partial M^{*\prime}}{a \partial \sin \Phi} = \frac{f \Omega a \cos^2 \Phi}{\overline{\cos^4 \phi}} (\cos^2 \phi)'. \tag{6.44}$$

On substituting (6.40) and (6.44) into the residual expression, the second and fourth terms disappear, yielding

$$\mathcal{R} = \frac{\kappa}{\mathsf{R}\bar{\rho}} \frac{\partial \bar{p}}{\partial \sin \Phi} \overline{p' \frac{\partial \sin \phi'}{a \partial \Lambda}} - \frac{f \Omega a \cos^2 \Phi}{\overline{\cos^4 \phi}} \frac{\partial \overline{\sin \phi}}{\partial S} \overline{p' \frac{\partial (\cos^2 \phi)'}{a \partial \Lambda}}. \tag{6.45}$$

Further simplification of this equation is obtained by deriving a mean thermal wind relation. Let us take a meridional derivative of the mean hydrostatic equation (6.39), take a vertical derivative of the mean gradient wind equation (6.43), and combine the resultant equations to obtain

$$\frac{\kappa}{\mathsf{R}\bar{\rho}} \frac{\partial \bar{p}}{a \cos \Phi \partial \Phi} = \frac{f \Omega a \cos^2 \Phi}{\overline{\cos^4 \phi}} \frac{\partial \overline{\cos^2 \phi}}{\partial S}. \tag{6.46}$$

We also note that the linearization of (3.50) yields

$$\overline{\sin\phi} = \sin\Phi + \frac{\overline{u_g \cos\phi}}{fa}, \tag{6.47}$$

and

$$\sin\phi' = \frac{(u_g \cos\phi)'}{fa}. \tag{6.48}$$

Substituting (6.46)–(6.48) into (6.45) and using (6.41) and (6.42), we finally obtain

$$\mathcal{R} = \frac{f\Omega a \cos^2\Phi}{\overline{\cos^4\phi}} \left[\frac{\partial \overline{\cos^2\phi}}{\partial S} \overline{p' \frac{\partial}{a\partial\Lambda} \left(\frac{(u_g\cos\phi)'}{fa} \right)} - \frac{\partial}{\partial S} \left(\frac{\overline{u_g\cos\phi}}{fa} \right) \overline{p' \frac{\partial(\cos^2\phi)'}{a\partial\Lambda}} \right] = 0. \tag{6.49}$$

With this result, the linearized invertibility principle can be written into the very simple form

$$f\cos\Phi \overline{v_g' \sigma^{*\prime}} = \nabla \cdot \mathbf{F}, \tag{6.50}$$

where $\nabla = (\partial/a\partial\Phi,\ \partial/\partial S)$ is the two dimensional del operator, and $\mathbf{F}$ is the combined geostrophic-gradient E-P flux in the meridional plane, taking the form

$$\mathbf{F} = \left(\frac{\partial \overline{(u_g\cos\phi)}}{\partial S} \overline{v_g' p'} - \frac{\partial \bar{p}}{\partial S} \overline{v_g'(u_g\cos\phi)'},\ \frac{\partial \bar{p}}{a\partial\Phi} \overline{v_g'(u_g\cos\phi)'} - [f\cos\Phi + \frac{\partial \overline{(u_g\cos\phi)}}{a\partial\Phi}] \overline{v_g' p'} \right). \tag{6.51}$$

On substituting (6.50) into the linearized potential pseudodensity equation (6.36), we obtain the generalized wave-activity relation for the mixed-balance system on the sphere developed in chapter 3:

$$\frac{\partial}{\partial T} \left[\tfrac{1}{2} \overline{\eta'^2} f^2 \cos\Phi \frac{\partial}{a\partial\Phi} \left(\frac{\bar{\sigma}^*}{f} \right) \right] + \nabla \cdot \mathbf{F} = 0. \tag{6.52}$$

For steady flow, this equation reduces to the Eliassen-Palm theorem. If the Eulerian mean of the particle displacement is somehow obtainable, diagnostic use of this equation provides the key step to solve a wave-mean flow interaction problem.

We shall next integrate (6.52) over the meridional-height plane. Let us first consider the vertical boundary conditions. We chose our top boundary as an isobaric surface, which implies

$$p' = 0, \quad \frac{\partial \bar{p}}{a\partial\Phi} = 0 \qquad \text{at} \quad S = S_T, \tag{6.53}$$

so that there is no E-P flux across the top boundary by referring to (6.51).

For the lower boundary, let us consider the basic-state version of (3.82b), i.e.,

$$c_p\bar{T} - \bar{M}^* + \tfrac{1}{2}\bar{u}_g^2 = 0 \qquad \text{at} \quad S = S_B, \tag{6.54}$$

which, after the using equation of state and the definition of entropy in the first term and taking the meridional derivative of the resultant equation, can be written as

$$\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial\Phi} - \frac{\partial\bar{M}^*}{\partial\Phi} + \bar{u}_g\frac{\partial\bar{u}_g}{\partial\Phi} = 0 \qquad \text{at} \quad S = S_B. \tag{6.55}$$

Note that the meridional derivative of the basic state version of the potential latitude formula yields $\partial\bar{u}_g/\partial\Phi = -fa\cos\Phi/\overline{\cos\phi} - \bar{\omega}a(\overline{\sin\phi})^{-1}\partial\overline{\cos\phi}/\partial\Phi$, where $\bar{\omega} = 2\Omega\overline{\sin\phi} + \bar{u}_g\overline{\tan\phi}/a$. Substituting this relation in (6.55) and using gradient wind equation, we obtain

$$\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial\Phi} - \frac{\bar{\omega}a\bar{u}_g}{\overline{\sin\phi}}\frac{\partial\overline{\cos\phi}}{\partial\Phi} = 0 \qquad \text{at} \quad S = S_B. \tag{6.56}$$

Next we consider the perturbation part of the lower boundary condition. Multiplying (3.82b) by $\cos^2\phi$ and linearizing it, we obtain

$$(c_p\bar{T} - \bar{M}^*)(\cos^2\phi)' + \overline{\cos^2\phi}c_pT' - \overline{\cos^2\phi}M^{*\prime} + \bar{u}_g\overline{\cos\phi}(u_g\cos\phi)' = 0 \quad \text{at} \quad S = S_B.$$

Multiplying this equation by v_g' and then taking a zonal average, we obtain

$$-\tfrac{1}{2}\bar{u}_g^2\,\overline{(\cos^2\phi)'u_g'} + \frac{1}{\bar{\rho}}\overline{\cos^2\phi}\;\overline{v_g'p'} - \frac{1}{f}\overline{\cos^2\phi}\;\overline{M^{*\prime}\frac{\partial M^{*\prime}}{a\cos\Phi\partial\Lambda}}$$
$$+(\overline{u_g\cos\phi})\;\overline{v_g'(u_g\cos\phi)'} = 0 \quad \text{at} \quad S = S_B, \tag{6.57}$$

in which we have also used the perturbed hydrostatic equation (6.40) and mean lower boundary condition (6.54). The third term in (6.57) vanishes after averaging. Using the potential radius relation (6.42), we can rewrite (6.57) as

$$\frac{\overline{\cos\phi}}{\bar{\rho}}\,\overline{v_g'p'} - \frac{\bar{\omega}\bar{u}_g}{2\Omega\overline{\sin\phi}}\,\overline{v_g'(u_g\cos\phi)'} = 0 \qquad \text{at} \quad S = S_B. \tag{6.58}$$

Now combining (6.56) and (6.58), we have proved that the E-P flux vanishes at the lower boundary, i.e.,

$$\frac{\partial\bar{p}}{a\partial\Phi}\overline{v_g'(u_g\cos\phi)'} - \left(f\cos\Phi + \frac{\partial(\overline{u_g\cos\phi})}{a\partial\Phi}\right)\overline{v_g'p'} = 0 \qquad \text{at} \quad S = S_B. \tag{6.59}$$

For lateral boundary conditions, we consider a channel bounded by two latitudes, and we assume that there is no perturbation meridional winds across both meridional walls, i.e.,

$$v_g' = 0 \qquad \text{at} \quad \Phi = \Phi_1,\ \Phi_2. \tag{6.60}$$

With all these boundary conditions, namely (6.53) and (6.59)–(6.60), we now can integrate (6.52) over the meridional plane, which yields

$$\frac{\partial}{\partial T}\iint \tfrac{1}{2}\overline{\eta'^2} f^2 \cos^2\Phi \frac{\partial}{a\partial\Phi}\left(\frac{\bar{\sigma}^*}{f}\right) ad\Phi dS = 0. \tag{6.61}$$

This, again, leads to the stability argument as follows: since the integral in (6.61) must be constant in time, in order for disturbances to grow in time, i.e., for $\overline{\eta'^2}$ to grow in time, the radial gradient of the basic state potential pseudodensity, $\partial\bar{\sigma}^*/a\partial\Phi$, must have both signs. Such a dynamical statement can be considered as a generalization of the Charney-Stern theorem for the three dimensional, mixed-balance system on the sphere.

Chapter 7

GENERAL BALANCED DYNAMICS FROM CLEBSCH POTENTIALS AND HAMILTON'S PRINCIPLE

The physical laws for fluid systems such as the atmosphere and ocean, discussed so far, are primarily based upon Newtonian mechanics. The beauty of the Newtonian dynamical system lies in its simple formulation, accessible to practical calculation and clear physical interpretation. Even so, in some theoretical analyses the Newtonian equations have been found to be unwieldy and inconvenient to use. In recent years, there has been a growing attention to applications of Hamiltonian mechanics to geophysical fluid systems (Benjamin 1984, Abarbanel *et al.* 1986, Salmon 1983, 1988, Shepherd 1990). Some progress has been made in the following two areas. The first of these applications is stability analysis. Fruitful theoretical results have been obtained in nonlinear stability problems, e.g., Arnol'd (1965, 1966), Abarbanel *et al.* (1986), McIntyre and Shepherd (1987) and Shepherd (1988, 1989), and generalization of wave-activity relations, e.g., Haynes (1988). The Casimir approach, convexity estimation and Liapunov stability criteria have played important roles in this class of studies. In the second class of applications, Hamilton's principle has been employed to search for a new set of dynamical governing equations. These studies include for example: finding the variable f semigeostrophic equations (Salmon 1985) and a planetary semigeostrophic theory (Shutts 1989), generalizing Eliassen's balanced vortex model (Craig 1991) and parameterizing geostrophic adjustment (Vallis 1992). The current chapter falls into this class of studies.

Some advantages of using Hamilton's approach to study a mechanical system have been cited many times (e.g., Lanczos 1970, Zhou 1978, Salmon 1983, Shepherd 1990). Two special advantages are extremely valuable for the second type of studies summarized above.

First, Hamilton's principle presents a remarkably succinct statement of the dynamics. In order to find new governing equations, one may merely concentrate on identifying a proper Hamiltonian or Lagrangian, from which all the Eulerian dynamical equations can naturally be derived by Hamilton's principle. Secondly, Noether's theorem, associated with the symmetries of the Hamiltonian, assures conservation principles for newly derived governing dynamical systems as long as the corresponding symmetries in these systems are preserved.

Before proceeding with our analysis, let us first briefly review some very basic concepts of Hamiltonian mechanics. We consider a finite-dimensional phase space $\{(q_i, \dot{q}_i)\}_{i=1,\dots,N}$ where q_i and $\dot{q}_i$ are the generalized coordinates and momenta respectively. The Lagrangian of the mechanical system is defined as the difference of the kinetic energy and potential energy, i.e.,

$$L(q,p) = \sum_i \tfrac{1}{2} m_i p_i \dot{q}_i - V(q_1, \dots, q_N), \tag{7.1}$$

where m_i is the mass of the ith particle, $V(q_i)$ is the potential energy, and p_i is the conjugate momentum which is related to the generalized momentum by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{7.2}$$

The Hamiltonian is defined as

$$H(q,p,\tau) = -L + \sum_i p_i \dot{q}_i, \tag{7.3}$$

which represents the total energy of the mechanical system if the kinetic energy is a homogeneous function of velocity to the second power (satisfying Euler's theorem). This is the case in most physical situations in fluid mechanics.

Hamilton's principle states that the virtual motion of a mechanical system from time τ_1 to time τ_2 is such that

$$\delta \int_{\tau_1}^{\tau_2} \left(\sum_i p_i \dot{q}_i - H \right) d\tau = 0, \tag{7.4}$$

where δ corresponds to independent variations of $p_i(\tau)$ and $q_i(\tau)$ and the variations at the end points are identically zero, i.e., $\delta q_i(\tau_1) = \delta q_i(\tau_2) = 0$.

By simultaneously taking variations of q_i and p_i, it is straightforward to show from (7.4) that Hamilton's canonical equations take the form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{7.5}$$

These equations provide important energy and cyclic symmetry properties which will be illustrated later.

In an infinite-dimensional (continuum) system, all the basic concepts discussed above for a discrete system are preserved by changing the summation over particles into an integral over volume. However, when one takes the Eulerian view and insists on a variational principle for the Eulerian dynamical system, the foregoing similarity between discrete and continuum systems is lost. This has been a long-standing difficulty in applying Hamiltonian mechanics to fluid systems since the Eulerian description is preferable in fluid dynamics. The solution to this difficulty has been found to be mysteriously related to the Clebsch velocity representation and Lin's constraint (Lin, 1963; Seliger and Whitham, 1968). Seliger and Whitham used these devices to construct a variational principle with a somewhat strange form from which an alternative set of Eulerian dynamical equations can be derived. Van Saarloos (1981) proved that Seliger and Whitham's variational principle for the Eulerian dynamical system can in fact be derived by a canonical transformation from the variational principle with Lagrangian description. In this chapter, we focus on constructing variational principles with the Lagrangian description, and deriving the Newtonian dynamical equations from such a traditionally formulated Hamilton's principle.

Another interesting aspect of the Clebsch velocity representation is that it serves as a set of canonical transformation coordinates to transform the hydrodynamical equations. By taking account of Lin's particle labeling coordinates, the Clebsch velocity decomposition is

$$\mathbf{v} = \nabla\chi + s\nabla\eta + \alpha_j\nabla\beta_j, \tag{7.6}$$

where s is entropy, α_j are the Lagrangian coordinates, and χ, η and β_j are scalar potentials. For quasi-static flows on an isentropic surface, the first and second terms in this expression can be combined so that vorticity is introduced only by the third term. Lamb

(1932, art. 167) and Schubert and Magnusdottir (1991) showed that this Clebsch representation of velocity can be used to transform the Eulerian dynamical equations to their canonical forms.

Following these studies, we show in the first section of this chapter how the Clebsch representation (7.6) results from vorticity coordinates, and how it can be used to transform the primitive equations of various forms (e.g., the shallow water equations, the quasi-static equations on an f-plane and the quasi-static equations on a sphere). In section 7.2, we introduce the variational principles for these different forms of primitive equations, and then using these variational principles coupled with the Clebsch representation as a set of variational constraints, we are able to derive the same canonical equations as those obtained in section 7.1. In the final section of this chapter, we extend the approach used in section 7.2 to balanced dynamics. In particular, we will first identify the variational principles for different balanced models. The approximated variational principle is then used, coupled with the reduced form of Clebsch representation, to produce a set of canonical equations in the balanced context. The goal is to shed some light on how a balanced model can be constructed in general and what the general structure of a balanced model should be.

7.1 Canonical transformation of the primitive equations by Clebsch potentials

In this section, we demonstrate that one of the physical interpretations of Clebsch potentials is closely related to vorticity coordinates, and two of these Clebsch potentials can be used as canonical coordinates to transform the dynamical system to its simplest mathematical form.

7.1.1 The shallow water primitive equations in cartesian coordinates

The shallow water equations on an f-plane can be written

$$\frac{\partial u}{\partial t} - \zeta v + \frac{\partial}{\partial x}[gh + \tfrac{1}{2}(u^2 + v^2)] = 0, \tag{7.7}$$

$$\frac{\partial v}{\partial t} + \zeta u + \frac{\partial}{\partial y}[gh + \tfrac{1}{2}(u^2 + v^2)] = 0, \tag{7.8}$$

$$\frac{Dh}{Dt}+h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0, \tag{7.9}$$

where u and v are the eastward and northward components of the velocity, h is the fluid depth,

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y} \tag{7.10}$$

the total derivative, and

$$\zeta=f+\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}, \tag{7.11}$$

the absolute vorticity.

Now consider a transformation from the coordinates (x,y,t) to the new coordinates $(X,Y,\mathcal{T})$, where $\mathcal{T}=t$. As with the notation in the previous chapters, the symbol $\mathcal{T}$ has been introduced to distinguish the time derivative at fixed (X,Y) from the time derivative at fixed (x,y). We require the new coordinates to be *vorticity coordinates* in the sense that the Jacobian of (X,Y) with respect to (x,y) is the dimensionless absolute vorticity, i.e.,

$$\zeta=f\frac{\partial(X,Y)}{\partial(x,y)}. \tag{7.12}$$

If we combine (7.11) and (7.12), the resulting expression can be rearranged into the form

$$\begin{aligned}&\frac{\partial}{\partial x}\left[v-\tfrac{1}{2}f(X-x)\left(\frac{\partial Y}{\partial y}+1\right)+\tfrac{1}{2}f(Y-y)\frac{\partial X}{\partial y}\right]\\&-\frac{\partial}{\partial y}\left[u-\tfrac{1}{2}f(X-x)\frac{\partial Y}{\partial x}+\tfrac{1}{2}f(Y-y)\left(\frac{\partial X}{\partial x}+1\right)\right]=0.\end{aligned} \tag{7.13}$$

Thus, the terms in the first brackets can be expressed as $\partial\chi/\partial y$ and the terms in the second brackets by $\partial\chi/\partial x$, where χ is a scalar potential. This results in

$$u=\frac{\partial\chi}{\partial x}+\tfrac{1}{2}f(X-x)\frac{\partial Y}{\partial x}-\tfrac{1}{2}f(Y-y)\left(\frac{\partial X}{\partial x}+1\right), \tag{7.14}$$

$$v=\frac{\partial\chi}{\partial y}+\tfrac{1}{2}f(X-x)\left(\frac{\partial Y}{\partial y}+1\right)-\tfrac{1}{2}f(Y-y)\frac{\partial X}{\partial y}. \tag{7.15}$$

Eqs. (7.14) and (7.15) can be regarded as Clebsch representations of the velocity field (Lamb 1932, page 248; Seliger and Whitham 1968), i.e., each component of a velocity vector can be expressed as the corresponding derivatives of the three scalar potentials χ, X and Y. Note the difference between these Clebsch representations and the Helmholtz decomposition of a vector velocity field, where velocity is split into the gradient of velocity potential and the curl of streamfunction. One criticism of using the Clebsch representation is that the physical interpretations of three scalar potentials are not clear. Here we see that the potential χ is purely related to the divergent part of the flow, since it vanishes when we take the curl of (7.14) and (7.15), as in the case of Helmholtz's decomposition. Part of X and Y is related to rotational flow field, indicated by the vorticity definition (7.12). Still, parts of X and Y also contribute to the irrotational flow field because the divergence of velocity field (7.14) and (7.15) involves not only χ but also X and Y. This is the essential difference between Helmholtz and Clebsch representations. As in Helmholtz's expression, rotational and irrotational flow fields are totally decoupled, while in Clebsch's expression the two are connected by the potentials X and Y. In this sense we may regard the Clebsch representation as a more general tool than the Helmholtz representation indealing with complex flows where linkage can occur between the irrotational and nondivergent components of the flow.

Another interpretation of (7.14) and (7.15), which may further aid the understanding of the Clebsch velocity potentials, results from noting that if $u - \partial\chi/\partial x$ and $v - \partial\chi/\partial y$ are approximated by their respective geostrophic wind components, and if $\partial Y/\partial x \approx 0$, $\partial X/\partial x \approx 1$, $\partial Y/\partial y \approx 1$ and $\partial X/\partial y \approx 0$, (7.14) and (7.15) in fact reduce to the geostrophic coordinates $X = x + v_g/f$ and $Y = y - u_g/f$. By this notion, we conjecture that the Clebsch representation may serve as a set of canonical coordinates which can be used to transform a dynamical equations to their simplest form. Since (7.14) and (7.15) are more complicated than the geostrophic coordinates, the dynamical equations to be transformed ought to be more general than the system with the geostrophic momentum approximation (GM), and this system must be the set of primitive equations (7.7) and (7.8).

To transform the original momentum equations we now take $\partial/\partial t$ of (7.14) and (7.15) to obtain

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[gh + \tfrac{1}{2}(u^2 + v^2)] = f\frac{\partial(X,Y)}{\partial(t,x)} + g\frac{\partial \mathcal{H}}{\partial x}, \tag{7.16}$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial y}[gh + \tfrac{1}{2}(u^2 + v^2)] = f\frac{\partial(X,Y)}{\partial(t,y)} + g\frac{\partial \mathcal{H}}{\partial y}, \tag{7.17}$$

where

$$g\mathcal{H} = gh + \tfrac{1}{2}(u^2+v^2) + \frac{\partial \chi}{\partial t} + \tfrac{1}{2}f(X-x)\frac{\partial Y}{\partial t} - \tfrac{1}{2}f(Y-y)\frac{\partial X}{\partial t}. \tag{7.18}$$

We now use (7.12) and the original equations (7.7) and (7.8) to rewrite (7.16) and (7.17) as

$$f\left(\frac{\partial Y}{\partial x}\frac{DX}{Dt} - \frac{\partial X}{\partial x}\frac{DY}{Dt}\right) + g\frac{\partial \mathcal{H}}{\partial x} = 0, \tag{7.19}$$

$$f\left(\frac{\partial Y}{\partial y}\frac{DX}{Dt} - \frac{\partial X}{\partial y}\frac{DY}{Dt}\right) + g\frac{\partial \mathcal{H}}{\partial y} = 0. \tag{7.20}$$

Together (7.19) and (7.20) imply that

$$(fV,\ -fU) = g\left(\frac{\partial \mathcal{H}}{\partial X},\ \frac{\partial \mathcal{H}}{\partial Y}\right), \tag{7.21}$$

where

$$U = \frac{DX}{Dt}, \qquad V = \frac{DY}{Dt}. \tag{7.22}$$

The first entry in (7.21) has been obtained by eliminating DX/Dt between (7.19) and (7.20), and the second entry by eliminating DY/Dt between (7.19) and (7.20). Thus, (7.21) represents the canonical shallow water equations transformed by the Clebsch velocity potential relation (7.14) and (7.15), and it presents a pair of geostrophic relations between the transformed winds U, V and the hyper-height $\mathcal{H}$. The terminology "hyper-height" used here simply restates the fact that given in (7.18), $\mathcal{H}$ is not just the physical height or Bernoulli height. It is the height which is further modified by a local time rate of change of the Clebsch potentials. From (7.18) and (7.21), one can see how the balanced dynamics derives from this generalized geostrophic balance relation. For quasi-geostrophic theory, the balance is achieved between the winds and the gradient of the first term on the right hand side of (7.18); for semigeostrophic theory, the balance is between the transformed winds and the gradient of the first term plus the geostrophic version of second term (or mixed geostrophic and gradient version of second term for mixed balanced theory). For unfiltered primitive equations, like (7.7)–(7.9), (7.21) no longer serves as a pure diagnostic relation between winds and pressure due to the transient information entering into this balanced relation.

We can easily show that (7.10) can also be written in $(X, Y, \mathcal{T})$ space as

$$\frac{D}{Dt} = \frac{\partial}{\partial \mathcal{T}} + U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y}. \tag{7.23}$$

The advantage of (7.23) over (7.10) is that the horizontal advecting velocity is expressed in terms of derivatives of $\mathcal{H}$ by (7.21), which are mathematically analogous to the geostrophic formulas.

The governing equation for the absolute vorticity can be derived from (7.7) and (7.8). It takes the form

$$\frac{D\zeta}{Dt} + \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{7.24}$$

Eliminating the divergence between (7.9) and (7.24) we obtain

$$\frac{DP}{Dt} = 0, \tag{7.25}$$

where $P = \zeta/h$ is the potential vorticity. Following the approach in Chapter 5, let us define the potential height as

$$h^* = \left(\frac{f}{\zeta}\right) h, \tag{7.26}$$

which indicates that the potential vorticity and the potential height are related by $Ph^* = f$, so that the potential height equation is

$$\frac{Dh^*}{Dt} = 0. \tag{7.27}$$

One should recall (e.g., Chapters 2 and 3) that in the case of balanced dynamics, (7.26) and (7.27) form a closed system, and this system represents the simplest mathematical formulation with one predictive equation and one invertibility principle. For the primitive equation model, however, due to the lack of a balance relation between the wind field ζ and the mass field h, (7.26) is uninvertable so that (2.26) and (2.27) do not form a closed system. Nevertheless, the foregoing Clebsch transformation points out a quite general way of thinking about balanced dynamics and how to obtain the simplest dynamical structure of a balanced model.

7.1.2 The quasi-static primitive equations in cylindrical coordinates

We now generalize the results obtained in the previous subsection to a fully stratified atmosphere. We also proceed in a cylindrical coordinate system in order to compare the results with the discussion in Chapter 2.

Using the entropy s as the vertical coordinate, we can write the quasi-static primitive equations on an f-plane as

$$\frac{\partial u}{\partial t} + \eta\dot{s} - \zeta v + \frac{\partial}{\partial r}[M + \tfrac{1}{2}(u^2 + v^2)] = 0, \tag{7.28}$$

$$\frac{\partial v}{\partial t} - \xi\dot{s} + \zeta u + \frac{\partial}{r\partial\phi}[M + \tfrac{1}{2}(u^2 + v^2)] = 0, \tag{7.29}$$

$$\frac{\partial M}{\partial s} = T, \tag{7.30}$$

$$\frac{D\sigma}{Dt} + \sigma\left(\frac{\partial(ru)}{r\partial r} + \frac{\partial v}{r\partial\phi} + \frac{\partial\dot{s}}{\partial s}\right) = 0, \tag{7.31}$$

with standard notations similar to those defined previously. Here the total derivative operator is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + v\frac{\partial}{r\partial\phi} + +\dot{s}\frac{\partial}{\partial s}, \tag{7.32}$$

and the three components of isentropic absolute vorticity are

$$(\xi,\ \eta,\ \zeta) = \left(-\frac{\partial v}{\partial s},\ \frac{\partial u}{\partial s},\ f + \frac{\partial(rv)}{r\partial r} - \frac{\partial u}{r\partial\phi}\right). \tag{7.33}$$

We shall now switch from the coordinates (r, ϕ, s, t) to the coordinates $(R, \Phi, S, \mathcal{T})$, where $S = s$ and $\mathcal{T} = t$. The symbols S and $\mathcal{T}$ are introduced to distinguish derivatives at fixed (R, Φ) from derivatives at fixed (r, ϕ). We require the new coordinates to be *vorticity coordinates* in the sense that

$$(\xi,\ \eta,\ \zeta) = f\left(\frac{\partial(\frac{1}{2}R^2, \Phi)}{r\partial(\phi, s)},\ \frac{\partial(\frac{1}{2}R^2, \Phi)}{\partial(s, r)},\ \frac{\partial(\frac{1}{2}R^2, \Phi)}{r\partial(r, \phi)}\right). \tag{7.34}$$

If we combine the third entries of (7.33) and (7.34), we obtain

$$f + \frac{\partial(rv)}{r\partial r} - \frac{\partial u}{r\partial\phi} = f\frac{\partial(\frac{1}{2}R^2, \Phi)}{r\partial(r, \phi)},$$

which can be rearranged into the form

$$\frac{\partial}{\partial r}\left[rv - \tfrac{1}{4}f(R^2 - r^2)\left(\frac{\partial\Phi}{\partial\phi} + 1\right) + \tfrac{1}{2}f(\Phi - \phi)R\frac{\partial R}{\partial\phi}\right]$$

$$-\frac{\partial}{\partial\phi}\left[u-\tfrac{1}{4}f(R^2-r^2)\frac{\partial\Phi}{\partial r}+\tfrac{1}{2}f(\Phi-\phi)\left(R\frac{\partial R}{\partial r}+r\right)\right]=0. \tag{7.35}$$

Thus, the terms in the first brackets can be expressed as $\partial\chi/\partial\phi$ and the terms in the second bracket as $\partial\chi/\partial r$, where χ is a scalar potential. This results in

$$u=\frac{\partial\chi}{\partial r}+\tfrac{1}{4}f(R^2-r^2)\frac{\partial\Phi}{\partial r}-\tfrac{1}{2}f(\Phi-\phi)\left(R\frac{\partial R}{\partial r}+r\right), \tag{7.36}$$

$$rv=\frac{\partial\chi}{\partial\phi}+\tfrac{1}{4}f(R^2-r^2)\left(\frac{\partial\Phi}{\partial\phi}+1\right)-\tfrac{1}{2}f(\Phi-\phi)R\frac{\partial R}{\partial\phi}. \tag{7.37}$$

Using (7.36) and (7.37) we can show that the first two entries of (7.34) are satisfied if

$$0=\frac{\partial\chi}{\partial s}+\tfrac{1}{4}f(R^2-r^2)\frac{\partial\Phi}{\partial s}-\tfrac{1}{2}f(\Phi-\phi)R\frac{\partial R}{\partial s}. \tag{7.38}$$

Equations (7.36)–(7.38) can be regarded as a Clebsch representation of the velocity field (Lamb 1932, page 248; Seliger and Whitham 1968). Compared with the shallow water version of this representation derived in the previous subsection, we have one more equation which is supposedly for the vertical component of velocity in the current stratified case. The zero on the left hand side (7.38), instead of the vertical velocity, is due to the fact that in the quasi-static system only the horizontal components of velocity appear in the right hand side of (7.33). In a more general nonhydrostatic argument (in the z-coordinate) the vertical velocity would contribute to the first two vorticity components in (7.33) and then would also appear on the left hand side of (7.38). Seliger and Whitham (1968) discuss how such Clebsch representations arise from variational principles. The zero on the left hand side of (7.38) can then be understood in terms of the neglect of the contribution of vertical motion to the kinetic energy in the Lagrangian for the quasi-static equations.

The physical interpretation of the three scalar potentials χ, R and Φ can be drawn in terms of the vorticity coordinates as discussed previously. There are two special cases of (7.36) and (7.37). First, when the flow is axisymmetric, i.e., $\partial\chi/\partial\phi=0$, $\partial R/\partial\phi=0$, and $\partial\Phi/\partial\phi=1$, then (7.37) reduces to $\frac{1}{2}fR^2=\frac{1}{2}fr^2+rv$. Since the right hand side of this expression is the absolute angular momentum per unit mass, in the axisymmetric case R is just an angular momentum coordinate and represents the radius to which a fluid parcel must be moved in order to change its tangential velocity to zero. This potential radius coordinate has been used by Schubert and Hack (1983) and Schubert and Alworth (1987) in studies of axisymmetric balanced tropical cyclones. In the second case, we approximate $u - \partial\chi/\partial r$ as geostrophic wind and $v - \partial\chi/\partial\phi$ by the gradient wind, and if $\partial\Phi/\partial r \approx 0$, $\partial R/\partial\phi \approx 0$, $\partial R/\partial r \approx 1$ and $\partial\Phi/\partial\phi \approx 1$ with R is not much different from r, (7.36) and (7.37) reduce to the set of combined geostrophic azimuth and potential radius coordinates, which has been used in Chapters 2 and 5. This suggests that (7.36) and (7.37) themselves are a set of generalized canonical coordinates, which may be used to transform a general dynamical system like (7.28)–(7.31).

To transform the original primitive equations (7.28)–(7.30) we now take $\partial/\partial t$ of (7.36)–(7.38) to obtain

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial r}[M + \tfrac{1}{2}(u^2 + v^2)] = f\frac{\partial(\frac{1}{2}R^2, \Phi)}{\partial(t, r)} + \frac{\partial \mathcal{M}}{\partial r}, \tag{7.39}$$

$$\frac{\partial(rv)}{\partial t} + \frac{\partial}{\partial\phi}[M + \tfrac{1}{2}(u^2 + v^2)] = f\frac{\partial(\frac{1}{2}R^2, \Phi)}{\partial(t, \phi)} + \frac{\partial \mathcal{M}}{\partial\phi}, \tag{7.40}$$

$$\frac{\partial}{\partial s}[M + \tfrac{1}{2}(u^2 + v^2)] = f\frac{\partial(\frac{1}{2}R^2, \Phi)}{\partial(t, s)} + \frac{\partial \mathcal{M}}{\partial s}, \tag{7.41}$$

where

$$\mathcal{M} = M + \tfrac{1}{2}(u^2 + v^2) + \frac{\partial\chi}{\partial t} + \tfrac{1}{4}f(R^2 - r^2)\frac{\partial\Phi}{\partial t} - \tfrac{1}{2}f(\Phi - \phi)R\frac{\partial R}{\partial t}. \tag{7.42}$$

Using (7.34) and the original momentum equations (7.28)–(7.30) we rewrite (7.39)–(7.41) as

$$fR\left(\frac{\partial\Phi}{\partial r}\frac{DR}{Dt} - \frac{\partial R}{\partial r}\frac{D\Phi}{Dt}\right) + \frac{\partial\mathcal{M}}{\partial r} = 0, \tag{7.43}$$

$$fR\left(\frac{\partial\Phi}{\partial\phi}\frac{DR}{Dt} - \frac{\partial R}{\partial\phi}\frac{D\Phi}{Dt}\right) + \frac{\partial\mathcal{M}}{\partial\phi} = 0, \tag{7.44}$$

$$fR\left(\frac{\partial\Phi}{\partial s}\frac{DR}{Dt} - \frac{\partial R}{\partial s}\frac{D\Phi}{Dt}\right) + \frac{\partial\mathcal{M}}{\partial s} = T. \tag{7.45}$$

Together (7.43)–(7.45) imply that

$$(fV, -fU, T) = \left(\frac{\partial\mathcal{M}}{\partial R}, \frac{\partial\mathcal{M}}{R\partial\Phi}, \frac{\partial\mathcal{M}}{\partial S}\right), \tag{7.46}$$

where

$$U = \frac{DR}{Dt}, \qquad V = R\frac{D\Phi}{Dt}. \tag{7.47}$$

The first entry in (7.46) has been obtained by eliminating $D\Phi/Dt$ between (7.43) and (7.44), the second entry by eliminating DR/Dt between (7.43) and (7.44), and the third entry by substituting the first two into (7.45). Thus, (7.46) represents the canonical quasi-static primitive equations transformed by Clebsch potentials (7.36)–(7.38), and it formally presents a pair of geostrophic relations and a hydrostatic relation.

The governing equation for the isentropic absolute vorticity can be derived from (7.28) and (7.29). It takes the form

$$\frac{D\zeta}{Dt} + \zeta\left(\frac{\partial(ru)}{r\partial r} + \frac{\partial v}{r\partial\phi}\right) = \left(\xi\frac{\partial}{\partial r} + \eta\frac{\partial}{r\partial\phi}\right)\dot{s}. \tag{7.48}$$

Eliminating the divergence between (7.31) and (7.48) we obtain

$$\frac{DP}{Dt} = \frac{1}{\sigma}\left(\xi\frac{\partial}{\partial r} + \eta\frac{\partial}{r\partial\phi} + \zeta\frac{\partial}{\partial s}\right)\dot{s}, \tag{7.49}$$

where $P = \zeta/\sigma$ is the Rossby-Ertel potential vorticity, and the total derivative operator can be written in $(R, \Phi, S, \mathcal{T})$ space as

$$\frac{D}{Dt} = \frac{\partial}{\partial\mathcal{T}} + U\frac{\partial}{\partial R} + V\frac{\partial}{R\partial\Phi} + \dot{S}\frac{\partial}{\partial S}. \tag{7.50}$$

Following the approach in Chapter 2, we now define the potential pseudodensity as

$$\sigma^* = \frac{f}{\zeta}\sigma, \tag{7.51}$$

which relates to the potential vorticity by $P\sigma^* = f$, so that we can write the predictive equation for potential pseudodensity in the form

$$\frac{\partial\sigma^*}{\partial\mathcal{T}} + \frac{\partial(\sigma^* RU)}{R\partial R} + \frac{\partial(\sigma^* V)}{R\partial\Phi} + \frac{\partial(\sigma^*\dot{S})}{\partial S} = 0. \tag{7.52}$$

Again, for balanced dynamics, (7.51) and (7.52) would form the simplest mathematical model.

7.1.3 The primitive equations in spherical coordinates

In the final part of this section, we will consider the most general case: a fully stratified fluid on a rotating sphere. The governing equations for this fluid motion can be written

$$\frac{\partial u}{\partial t} + \eta\dot{s} - \zeta v + \frac{\partial}{a\cos\phi\partial\lambda}[M + \tfrac{1}{2}(u^2 + v^2)] = 0, \tag{7.53}$$

$$\frac{\partial v}{\partial t} - \xi\dot{s} + \zeta u + \frac{\partial}{a\partial\phi}[M + \tfrac{1}{2}(u^2+v^2)] = 0, \tag{7.54}$$

$$\frac{\partial M}{\partial s} = T, \tag{7.55}$$

$$\frac{D\sigma}{Dt} + \sigma\left(\frac{\partial u}{a\cos\phi\partial\lambda} + \frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial\dot{s}}{\partial s}\right) = 0. \tag{7.56}$$

All the notation is standard and can be found in Chapter 3. The total derivative is in the form

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{a\cos\phi\partial\lambda} + v\frac{\partial}{a\partial\phi} + \dot{s}\frac{\partial}{\partial s}, \tag{7.57}$$

and the three components of isentropic absolute vorticity are

$$(\xi,\ \eta,\ \zeta) = \left(-\frac{\partial v}{\partial s},\ \frac{\partial u}{\partial s},\ 2\Omega\sin\phi + \frac{\partial v}{a\cos\phi\partial\lambda} - \frac{\partial(u\cos\phi)}{a\cos\phi\partial\phi}\right). \tag{7.58}$$

Let us now transform this dynamical system from (λ,ϕ,s,t) space to $(\Lambda,\Phi,S,\mathcal{T})$ space. We require the new coordinates to be *vorticity coordinates* in the sense that

$$(\xi,\ \eta,\ \zeta) = 2\Omega\sin\Phi\left(\frac{\partial(\Lambda,\sin\Phi)}{\partial(\phi,s)},\ \frac{\partial(\Lambda,\sin\Phi)}{\cos\phi\partial(s,\lambda)},\ \frac{\partial(\Lambda,\sin\Phi)}{\partial(\lambda,\sin\phi)}\right). \tag{7.59}$$

If we combine the third entries of (7.58) and (7.59), we obtain

$$2\Omega\sin\phi + \frac{\partial v}{a\cos\phi\partial\lambda} - \frac{\partial(u\cos\phi)}{a\cos\phi\partial\phi} = 2\Omega\sin\Phi\frac{\partial(\Lambda,\sin\Phi)}{\partial(\lambda,\sin\phi)},$$

which can be rearranged into the form

$$\begin{aligned}&\frac{\partial}{\partial\lambda}\left[v + \tfrac{1}{2}\Omega a(\sin^2\Phi - \sin^2\phi)\frac{\partial\Lambda}{\partial\phi} - \tfrac{1}{2}\Omega a(\Lambda-\lambda)\frac{\partial(\sin^2\Phi+\sin^2\phi)}{\partial\phi}\right]\\ &-\frac{\partial}{\partial\phi}\left[u\cos\phi + \tfrac{1}{2}\Omega a(\sin^2\Phi - \sin^2\phi)\frac{\partial(\Lambda+\lambda)}{\partial\lambda} - \tfrac{1}{2}\Omega a(\Lambda-\lambda)\frac{\partial(\sin^2\Phi)}{\partial\lambda}\right] = 0.\end{aligned} \tag{7.60}$$

Thus, the terms in first brackets can be expressed as $\partial\chi/a\partial\phi$ and the terms in second brackets as $\partial\chi/a\partial\lambda$, where χ is a scalar potential. This results in

$$u\cos\phi = \frac{\partial\chi}{a\partial\lambda} - \tfrac{1}{2}\Omega a(\sin^2\Phi - \sin^2\phi)\frac{\partial(\Lambda+\lambda)}{\partial\lambda} + \tfrac{1}{2}\Omega a(\Lambda-\lambda)\frac{\partial(\sin^2\Phi)}{\partial\lambda}, \tag{7.61}$$

$$v = \frac{\partial\chi}{a\partial\phi} - \tfrac{1}{2}\Omega a(\sin^2\Phi - \sin^2\phi)\frac{\partial\Lambda}{\partial\phi} + \tfrac{1}{2}\Omega a(\Lambda-\lambda)\frac{\partial(\sin^2\Phi+\sin^2\phi)}{\partial\phi}. \tag{7.62}$$

Using (7.61) and (7.62) we can show that the first two entries of (7.59) are satisfied if

$$0 = \frac{\partial\chi}{\partial s} - \tfrac{1}{2}\Omega a^2(\sin^2\Phi - \sin^2\phi)\frac{\partial\Lambda}{\partial s} + \tfrac{1}{2}\Omega a^2(\Lambda-\lambda)\frac{\partial(\sin^2\Phi)}{\partial s}. \tag{7.63}$$

Equations (7.61)–(7.63) again constitute Clebsch representations of the velocity field. The zero on the left hand side of (7.63) is due to the quasi-static approximation made in the original governing equations, which has been discussed in the previous subsection.

Several special cases are noted from (7.61) and (7.62). (1) When the flow is zonally symmetric (i.e., $\partial\chi/\partial\lambda = 0$, $\partial\Phi/\partial\lambda = 0$ and $\partial\Lambda/\partial\lambda = 1$), (7.61) reduces to $\Omega a \sin^2\Phi = \Omega a \sin^2\phi - u\cos\phi$, or equivalently $\Omega a^2\cos^2\Phi = \Omega a^2\cos^2\phi + ua\cos\phi$, the right hand side of which is the absolute angular momentum per unit mass. Thus, in the zonally symmetric case, Φ is an angular momentum coordinate and represents the latitude to which a parcel must be moved in order to change its zonal velocity to zero. This potential latitude coordinate has proved useful in studies of the ITCZ and the Hadley circulation (Hack *et al.* 1989, Schubert *et al.* 1991). (2) When we approximate $\sin^2\Phi \approx \sin^2\phi$ in (7.62) and $\Lambda \approx \lambda$ in (7.61), and if $u - \partial\chi/a\cos\phi\partial\lambda$ is defined as the gradient zonal wind and $v - \partial\chi/a\partial\phi$ is defined as the geostrophic meridional wind, (7.61) and (7.62) reduce to potential latitude and geostrophic longitude coordinates respectively discussed in Chapter 3 [(3.50) and (3.51)]. This coordinate set is crucial in constructing the mixed geostrophic gradient balanced theory. (3) With the approximations made in (2), we further assume that $\sin\Phi \approx \sin\phi$ in (7.61), and define $u - \partial\chi/a\cos\phi\partial\lambda$ as the geostrophic zonal wind. (7.61) and (7.62) then reduce to a pair of generalized geostrophic coordinates, which has been used by Magnusdottir and Schubert (1991) to construct semigeostrophic theory on the sphere.

Let us now use (7.61)–(7.63) to transform the original primitive equations (7.53)–(7.55). The procedures are exactly the same as before. We take $\partial/\partial t$ of (7.61)–(7.63), which yields

$$\frac{\partial(u\cos\phi)}{\partial t} + \frac{\partial}{a\partial\lambda}[M + \tfrac{1}{2}(u^2+v^2)] = a2\Omega\sin\Phi\frac{\partial(\Lambda,\sin\Phi)}{\partial(t,\lambda)} + \frac{\partial\mathcal{M}}{a\partial\lambda}, \tag{7.64}$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{a\partial\phi}[M + \tfrac{1}{2}(u^2+v^2)] = a2\Omega\sin\Phi\frac{\partial(\Lambda,\sin\Phi)}{\partial(t,\phi)} + \frac{\partial\mathcal{M}}{a\partial\phi}, \tag{7.65}$$

$$\frac{\partial}{\partial s}[M + \tfrac{1}{2}(u^2+v^2)] = a^2 2\Omega\sin\Phi\frac{\partial(\Lambda,\sin\Phi)}{\partial(t,s)} + \frac{\partial\mathcal{M}}{\partial s}, \tag{7.66}$$

where

$$\mathcal{M} = M + \tfrac{1}{2}(u^2+v^2) + \frac{\partial\chi}{\partial t} - \tfrac{1}{2}\Omega a^2(\sin^2\Phi - \sin^2\phi)\frac{\partial\Lambda}{\partial t} + \tfrac{1}{2}\Omega a^2(\Lambda - \lambda)\frac{\partial(\sin^2\Phi)}{\partial t}. \tag{7.67}$$

Using (7.59) and the original momentum equations (7.53)–(7.55) we rewrite (7.64)–(7.66) as

$$2\Omega \sin\Phi \left(\frac{\partial\Phi}{\partial\lambda}\frac{D\Lambda}{Dt} - \frac{\partial\Lambda}{\partial\lambda}\frac{D\Phi}{Dt}\right) a^2\cos\Phi + \frac{\partial\mathcal{M}}{\partial\lambda} = 0, \tag{7.68}$$

$$2\Omega \sin\Phi \left(\frac{\partial\Phi}{\partial\phi}\frac{D\Lambda}{Dt} - \frac{\partial\Lambda}{\partial\phi}\frac{D\Phi}{Dt}\right) a^2\cos\Phi + \frac{\partial\mathcal{M}}{\partial\phi} = 0, \tag{7.69}$$

$$2\Omega \sin\Phi \left(\frac{\partial\Phi}{\partial s}\frac{D\Lambda}{Dt} - \frac{\partial\Lambda}{\partial s}\frac{D\Phi}{Dt}\right) a^2\cos\Phi + \frac{\partial\mathcal{M}}{\partial s} = T. \tag{7.70}$$

Together (7.68)–(7.70) imply that

$$(2\Omega V\sin\Phi,\ -2\Omega U\sin\Phi,\ T) = \left(\frac{\partial\mathcal{M}}{a\cos\Phi\partial\Lambda}, \frac{\partial\mathcal{M}}{a\partial\Phi}, \frac{\partial\mathcal{M}}{\partial S}\right), \tag{7.71}$$

where

$$U = a\cos\Phi\frac{D\Lambda}{Dt}, \qquad V = a\frac{D\Phi}{Dt}. \tag{7.72}$$

The first entry in (7.71) has been obtained by eliminating $D\Lambda/Dt$ between (7.68) and (7.69), and the second entry by eliminating $D\Phi/Dt$ between (7.68) and (7.69), and the third entry by substituting the first two into (7.70). Thus, (7.71) represents the canonical quasi-static primitive equations transformed by Clebsch velocity potentials (7.61)–(7.63), and it formally resembles a pair of geostrophic relations and a hydrostatic balanced relation.

The governing equation for the isentropic absolute vorticity can be derived from (7.53) and (7.54), or equivalently from (7.71) and (7.72). In either case it takes the form

$$\frac{D\zeta}{Dt} + \zeta\left(\frac{\partial u}{a\cos\phi\partial\lambda} + \frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi}\right) = \left(\xi\frac{\partial}{\partial\lambda} + \eta\frac{\partial}{\partial\phi}\right)\dot{s}. \tag{7.73}$$

Eliminating the divergence between (7.56) and (7.73) we obtain

$$\frac{DP}{Dt} = \frac{1}{\sigma}\left(\xi\frac{\partial}{\partial\lambda} + \eta\frac{\partial}{\partial\phi} + \zeta\frac{\partial}{\partial s}\right)\dot{s} = \zeta\frac{\partial\dot{s}}{\partial S}, \tag{7.74}$$

where $P = \zeta/\sigma$ is the Rossby-Ertel potential vorticity. The total derivative operator can be written in $(\Lambda, \Phi, S, \mathcal{T})$ space as

$$\frac{D}{Dt} = \frac{\partial}{\partial\mathcal{T}} + U\frac{\partial}{a\cos\Phi\partial\Lambda} + V\frac{\partial}{a\partial\Phi} + \dot{S}\frac{\partial}{\partial S}. \tag{7.75}$$

Following the approach in Chapter 3, we now define the potential pseudodensity as

$$\sigma^* = \frac{2\Omega \sin \Phi}{\zeta} \sigma. \tag{7.76}$$

The potential vorticity and the potential pseudodensity are related by $P\sigma^* = 2\Omega \sin \Phi$. The potential pseudodensity equation can be easily obtained from the potential vorticity equation (7.74), and its flux form is particularly convenient. With D/Dt given by (7.75), this flux form can be written

$$\frac{\partial \sigma^*}{\partial T} + \frac{\partial(\sigma^* U)}{a \cos \Phi \partial \Lambda} + \frac{\partial(\sigma^* V \cos \Phi)}{a \cos \Phi \partial \Phi} + \frac{\partial(\sigma^* \dot{S})}{\partial S} = 0. \tag{7.77}$$

Again, for balanced dynamics, (7.76) and (7.77) would form the simplest mathematical model.

7.2 Hamiltonian structure of the primitive equations and their canonical transformation

In the last section we have shown that the Clebsch potentials are related to a set of vorticity coordinates which can be used to transform dynamical equations of Eulerian form to their simplest mathematical form. We now introduce this idea to the Hamiltonian systems associated with the primitive equation models discussed above. We first identify the variational principles for the primitive equation systems by deriving the set of Eulerian equations from such principles, and then demonstrate how conservation principles are associated with the symmetries of the Lagrangians. Finally, we conduct independent variations in the transformed phase space with Clebsch representation as a set of constraints. Such operations surprisingly lead to the canonical equations obtained in the previous section, suggesting a general way to construct a new dynamical system from a proper Hamilton principle with corresponding Clebsch potentials.

7.2.1 The shallow water primitive equations in cartesian coordinates

Following the Lagrangian description, the position of a fluid particle is determined by its labeling coordinates x_0, y_0 and time τ, i.e.,

$$x = x(x_0, y_0, \tau), \qquad y = y(x_0, y_0, \tau).$$

At time τ, the mass of a marked fluid parcel is related to its initial mass by

$$d(\text{mass}) = \rho h dx dy = h_0 dx_0 dy_0, \tag{7.78}$$

which can be written in the Jacobian form

$$\frac{h_0}{\rho h} = \frac{\partial(x,y)}{\partial(x_0,y_0)}, \tag{7.79}$$

where ρ is the constant fluid density, h_0 and h are the fluid surface heights at the initial time and at time τ respectively.

The time rate of change of (7.79) is as follows

$$\frac{\partial}{\partial\tau}\left(\frac{h_0}{\rho h}\right) = -\frac{h_0}{\rho h^2}\frac{\partial h}{\partial \tau} = \frac{\partial}{\partial\tau}\left[\frac{\partial(x,y)}{\partial(x_0,y_0)}\right] = \frac{\partial(\dot{x},y)}{\partial(x_0,y_0)} + \frac{\partial(x,\dot{y})}{\partial(x_0,y_0)},$$

where the first term on the right hand side of this expression is

$$\frac{\partial(\dot{x},y)}{\partial(x_0,y_0)} = \frac{\partial(\dot{x},y)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(x_0,y_0)} = \frac{h_0}{\rho h}\frac{\partial u}{\partial x},$$

and similarly the second term is

$$\frac{\partial(x,\dot{y})}{\partial(x_0,y_0)} = \frac{\partial(x,\dot{y})}{\partial(x,y)}\frac{\partial(x,y)}{\partial(x_0,y_0)} = \frac{h_0}{\rho h}\frac{\partial v}{\partial y}.$$

Combining all these results, we have

$$\frac{\partial h}{\partial\tau} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{7.80}$$

Note that in the Lagrangian system, the time rate of change is in the sense of following the particle motion. Hereafter we use the notation $\partial/\partial\tau = D/Dt$ as the total time derivative, and $\partial/\partial t$ as the local time derivative. Therefore, we have derived the mass continuity eqation (7.80) through the Lagrangian particle labeling system.

Following Salmon (1983, 1985), we define the Lagrangian for the shallow water primitive system as

$$L = \iint \left[(u - \Omega y)\frac{\partial x}{\partial\tau} + (v + \Omega x)\frac{\partial y}{\partial \tau}\right] h_0 dx_0 dy_0 - H \tag{7.81}$$

where

$$H = \iint \tfrac{1}{2}(u^2 + v^2 + gh) h_0 dx_0 dy_0, \tag{7.82}$$

so that the principle governing the fluid motion in a shallow water system on an f-plane can be stated as

$$\delta \int L d\tau = 0. \tag{7.83}$$

We can prove that (7.81)–(7.83) form the correct Hamilton principle for the shallow water primitive equation model (7.7)–(7.9). To see this, let us calculate (7.83) by taking the independent variation $\delta\mathbf{x} = (\delta x, \delta y)$ of it, which yields

δx:

$$\begin{aligned}
&\int d\tau \iint \left[(u - \Omega y)\frac{\partial(\delta x)}{\partial\tau} + \Omega\delta x\frac{\partial y}{\partial\tau} - \tfrac{1}{2}g\delta h\right] h_0 dS_0 \\
= &\int d\tau \iint \left[-\frac{\partial}{\partial\tau}(u - \Omega y) + \Omega\frac{\partial y}{\partial\tau} - \tfrac{1}{2}\rho g\frac{\partial(h^2, y)}{\partial(x_0, y_0)}\right] \delta x h_0 dS_0 = 0,
\end{aligned}$$

where $dS_0 = dx_0 dy_0$ stands for the area integral-element. The second line is obtained using integration by parts with vanishing of variations at endpoints. Since the variation δx is arbitrary, the terms inside the brackets must be zero, which gives

$$\frac{\partial u}{\partial\tau} - fv + g\frac{\partial h}{\partial x} = 0, \tag{7.84}$$

where $f = 2\Omega$. Similarly, the variation in y is

δy:

$$\begin{aligned}
&\int d\tau \iint \left[-\Omega\delta y\frac{\partial x}{\partial\tau} + (v + \Omega x)\frac{\partial(\delta y)}{\partial\tau} - \tfrac{1}{2}g\delta h\right] h_0 dS_0 \\
= &\int d\tau \iint \left[-\Omega\frac{\partial x}{\partial\tau} - \frac{\partial}{\partial\tau}(v + \Omega x) - \tfrac{1}{2}\rho g\frac{\partial(x, h^2)}{\partial(x_0, y_0)}\right] \delta y h_0 dS_0 = 0,
\end{aligned}$$

which yields

$$\frac{\partial v}{\partial\tau} + fu + g\frac{\partial h}{\partial y} = 0. \tag{7.85}$$

Collecting (7.80), (7.84) and (7.85), we have derived the shallow water primitive equation system from Hamilton's principle (7.81)–(7.83).

The variation of (7.81) in the other two coordinates of phase space yields

δu:

$$\int d\tau \iint \left(\frac{\partial x}{\partial\tau} - u\right) \delta u h_0 dS_0 = 0,$$

which just gives the definition of u-velocity: $u = \partial x/\partial \tau$. Similarly, δv of (7.81) will give the definition of v-velocity: $v = \partial y/\partial \tau$.

We next discuss how the conservation principles result from the governing equations (7.81)–(7.83). We first note that the Hamiltonian, the total energy of the system, is a function in phase space such that $H = H(x, y, u, v, t)$. Simultaneously taking variations δx, δy, δu, δv and δt of (7.81), we have

$$\int d\tau \iint \Big[(u-\Omega y)\frac{\partial(\delta x)}{\partial \tau} + (\delta u - \Omega \delta y)\frac{\partial x}{\partial \tau} + (v+\Omega x)\frac{\partial(\delta y)}{\partial \tau} + (\delta v + \Omega \delta x)\frac{\partial y}{\partial \tau}$$
$$-\frac{\partial H^*}{\partial x}\delta x - \frac{\partial H^*}{\partial y}\delta y - \frac{\partial H^*}{\partial u}\delta u - \frac{\partial H^*}{\partial v}\delta v - \frac{\partial H^*}{\partial t}\delta t\Big]\, h_0 dS_0 = 0,$$

where H^* stands for the integrand of H, which is the sum of the kinetic energy and surface potential energy per unit mass. That the variations are independent and arbitrary leads to the following canonical equations:

$$\frac{\partial H^*}{\partial x} = -\dot{u} + f\dot{y}, \quad \frac{\partial H^*}{\partial y} = -\dot{v} - f\dot{x},$$
$$\frac{\partial H^*}{\partial u} = \dot{x}, \quad \frac{\partial H^*}{\partial v} = -\dot{y}, \quad \frac{\partial H^*}{\partial t} = 0.$$

On substituting these relations into the total derivative of the Hamiltonian,

$$\frac{dH^*}{dt} = \frac{\partial H^*}{\partial t} + \frac{\partial H^*}{\partial x}\dot{x} + \frac{\partial H^*}{\partial y}\dot{y} + \frac{\partial H^*}{\partial u}\dot{u} + \frac{\partial H^*}{\partial v}\dot{v}$$

one obtains a statement of energy conservation:

$$\frac{dH^*}{dt} = 0. \tag{7.86}$$

From this derivation we see that the existence of an energy invariant is closely related to the symmetries in the Hamiltonian structure and the fact that time does not explicitly appear in the Hamiltonian.

Salmon (1983) points out that the conservation of potential vorticity in a Hamiltonian system is due to the existence of a special kind of symmetry, the so called "particle relabeling symmetry", in the flow field. Let us take variations of (7.81)–(7.83) in the labeling coordinates, i.e., $\delta(x_0, y_0)$, while holding (x, y) fixed. In so doing, we have

$$\int d\tau \iint \left[(u-\Omega y)\delta\left(\frac{\partial x}{\partial \tau}\right) + (v+\Omega x)\delta\left(\frac{\partial y}{\partial \tau}\right)\right] h_0 dS_0 = 0. \tag{7.87}$$

Using the chain rule, we can prove the following relations:

$$\delta\left(\frac{\partial x}{\partial \tau}\right) = -\frac{\partial x}{\partial x_0}\frac{\partial(\delta x_0)}{\partial \tau} - \frac{\partial x}{\partial y_0}\frac{\partial(\delta y_0)}{\partial \tau}, \tag{7.88}$$

$$\delta\left(\frac{\partial y}{\partial \tau}\right) = -\frac{\partial y}{\partial x_0}\frac{\partial(\delta x_0)}{\partial \tau} - \frac{\partial y}{\partial y_0}\frac{\partial(\delta y_0)}{\partial \tau}. \tag{7.89}$$

The variation of (7.79) in the labeling coordinates, i.e.,

$$0 = \delta\left[\frac{\partial(x_0, y_0)}{\partial(x, y)}\right] = \frac{\partial(\delta x_0, y_0)}{\partial(x, y)} + \frac{\partial(x_0, \delta y_0)}{\partial(x, y)} = \frac{\rho h}{h_0}\left(\frac{\partial \delta x_0}{\partial x_0} + \frac{\partial \delta y_0}{\partial y_0}\right)$$

leads to

$$\delta x_0 = -\frac{\partial \delta\psi}{\partial y_0}, \qquad \delta y_0 = \frac{\partial \delta\psi}{\partial x_0}. \tag{7.90}$$

Substituting in (7.88)–(7.90), (7.87) becomes

$$\begin{aligned}
&-\int d\tau \iint \left\{\left[(u-\Omega y)\frac{\partial x}{\partial x_0} + (v+\Omega x)\frac{\partial y}{\partial x_0}\right]\frac{\partial \delta x_0}{\partial \tau} + \left[(u-\Omega y)\frac{\partial x}{\partial y_0}\right.\right.\\
&\qquad\qquad \left.\left. +(v+\Omega x)\frac{\partial y}{\partial y_0}\right]\frac{\partial \delta y_0}{\partial \tau}\right\} h_0 dS_0\\
&= \int d\tau \iint \left\{\delta x_0 \frac{\partial}{\partial \tau}\left[(u-\Omega y)\frac{\partial x}{\partial x_0} + (v+\Omega x)\frac{\partial y}{\partial x_0}\right] + \delta y_0 \frac{\partial}{\partial \tau}\left[(u-\Omega y)\frac{\partial x}{\partial y_0}\right.\right.\\
&\qquad\qquad \left.\left. +(v+\Omega x)\frac{\partial y}{\partial y_0}\right]\right\} h_0 dS_0\\
&= \int d\tau \iint \delta\psi \frac{\partial}{\partial \tau}\left[\frac{\partial(x, u-\Omega y)}{\partial(x_0, y_0)} + \frac{\partial(y, v+\Omega x)}{\partial(x_0, y_0)}\right] h_0 dS_0 = 0.
\end{aligned}$$

Then due to the arbitrariness of $\delta\psi$, we obtain

$$\frac{\partial}{\partial \tau}\left[\left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{1}{h}\right] = 0, \tag{7.91}$$

which indicates that the potential vorticity is conserved on particles.

We now consider a phase space that is spanned by two physical coordinates (x, y) and two Clebsch potentials (X, Y). In the previous section we have shown that X and Y are in fact the two vorticity coordinates, and the Eulerian form of the primitive equations can be transformed to the simplest form by such coordinates. Since the horizontal velocity field can be expressed by these potentials, we may take the independent variations of (7.81)–(7.83) in X and Y subject to the constraints (7.14) and (7.15). Let us first write the Lagrangian (7.81) in the form

$$L = \iint \left[(u-\Omega y)\frac{\partial x}{\partial \tau} + (v+\Omega x)\frac{\partial y}{\partial \tau} - g\mathcal{H} + \frac{\partial \chi}{\partial t} + \tfrac{1}{2}f(X-x)\frac{\partial Y}{\partial t}\right.$$

$$-\tfrac{1}{2}f(Y-y)\frac{\partial X}{\partial t}\Big]\, h_0 dS_0, \tag{7.92}$$

where $\mathcal{H}$ is defined in (7.18).

We then take a variation δX of (7.14) and (7.15), resulting in

$$\delta u = \tfrac{1}{2}f\frac{\partial Y}{\partial x}\delta X - \tfrac{1}{2}f(Y-y)\frac{\partial(\delta X)}{\partial x}, \tag{7.93}$$

$$\delta v = \tfrac{1}{2}f\left(\frac{\partial Y}{\partial y}+1\right)\delta X - \tfrac{1}{2}f(Y-y)\frac{\partial(\delta X)}{\partial y}. \tag{7.94}$$

Using these two relations and taking the variation of (7.92), we have

δX:

$$\begin{aligned}
&\int d\tau \iint \left[\frac{\partial x}{\partial \tau}\delta u + \frac{\partial y}{\partial \tau}\delta v + \tfrac{1}{2}f\frac{\partial Y}{\partial t}\delta X - \tfrac{1}{2}f(Y-y)\frac{\partial(\delta X)}{\partial t} - g\frac{\partial \mathcal{H}}{\partial X}\delta X\right] h_0 dS_0\\
=\;& \int d\tau \iint \Big[\tfrac{1}{2}fh_0 u\frac{\partial Y}{\partial x} + \frac{\partial}{\partial x}\left(\tfrac{1}{2}fh_0 u(Y-y)\right) + \tfrac{1}{2}fh_0 v\left(\frac{\partial Y}{\partial y}+1\right)\\
&+\frac{\partial}{\partial y}\left(\tfrac{1}{2}fh_0 v(Y-y)\right) + \tfrac{1}{2}fh_0\frac{\partial Y}{\partial t} + \frac{\partial}{\partial t}\left(\tfrac{1}{2}fh_0(Y-y)\right) - g\frac{\partial \mathcal{H}}{\partial X}h_0\Big]\,\delta X dS_0 = 0,
\end{aligned}$$

which leads to

$$f\frac{DY}{Dt} = g\frac{\partial \mathcal{H}}{\partial X}, \tag{7.95}$$

by using the continuity equation.

Similarly, when we take a variation δY of (7.14) and (7.15), we get

$$\delta u = \tfrac{1}{2}f(X-x)\frac{\partial(\delta Y)}{\partial x} - \tfrac{1}{2}f\left(\frac{\partial X}{\partial x}+1\right)\delta Y, \tag{7.96}$$

$$\delta v = \tfrac{1}{2}f(X-x)\frac{\partial(\delta Y)}{\partial y} - \tfrac{1}{2}f\frac{\partial X}{\partial y}\delta Y, \tag{7.97}$$

and therefore the variation of (7.92) can be calculated as

δY:

$$\begin{aligned}
&\int d\tau \iint \left[\frac{\partial x}{\partial \tau}\delta u + \frac{\partial y}{\partial \tau}\delta v - \tfrac{1}{2}f\frac{\partial X}{\partial t}\delta Y + \tfrac{1}{2}f(X-x)\frac{\partial(\delta Y)}{\partial t} - g\frac{\partial \mathcal{H}}{\partial Y}\delta Y\right] h_0 dS_0\\
=\;& \int d\tau \iint \Big[-\tfrac{1}{2}fh_0 u\left(\frac{\partial X}{\partial x}+1\right) - \frac{\partial}{\partial x}\left(\tfrac{1}{2}fh_0 u(X-x)\right) - \tfrac{1}{2}fh_0 v\frac{\partial X}{\partial y}\\
&-\frac{\partial}{\partial y}\left(\tfrac{1}{2}fh_0 v(X-x)\right) - \tfrac{1}{2}fh_0\frac{\partial X}{\partial t} - \frac{\partial}{\partial t}\left(\tfrac{1}{2}fh_0(X-x)\right) - g\frac{\partial \mathcal{H}}{\partial Y}h_0\Big]\,\delta Y dS_0 = 0,
\end{aligned}$$

which leads to

$$-f\frac{DX}{Dt} = g\frac{\partial \mathcal{H}}{\partial Y}. \tag{7.98}$$

Equations (7.95) and (7.98) are the main results for this subsection, and they should be compared with (7.21) of the previous section. The idea that we are trying to illustrate here is that from a variational principle in the Lagrangian description plus Clebsch representations of velocity, we can obtain a set of canonical momentum equations that are identical to those obtained by directly transforming the Eulerian equations using Clebsch representations.

7.2.2 The quasi-static primitive equations in cylindrical coordinates

For the fully stratified fluid system there is one more dimension added to the problem so that the position of fluid parcel is, in the cylindrical coordinates,

$$r = r(r_0, \phi_0, s_0, \tau), \ \ \phi = \phi(r_0, \phi_0, s_0, \tau), \ \ s = s(r_0, \phi_0, s_0, \tau)$$

and the mass continuity can be expressed as

$$d(\text{mass}) = \sigma r dr d\phi ds = \sigma_0 r_0 dr_0 d\phi_0 ds_0, \tag{7.99}$$

which can be written in the Jacobian form

$$\frac{\sigma_0}{\sigma} = \frac{\partial(\frac{1}{2}r^2, \phi, s)}{\partial(\frac{1}{2}r_0^2, \phi_0, s_0)}, \tag{7.100}$$

where σ and σ_0 are the pseudodensity at time τ and at the initial time, respectively. In the z-coordinate system, the proper form of (7.100) is

$$\frac{\rho_0}{\rho} = \frac{\partial(\frac{1}{2}r^2, \phi, z)}{\partial(\frac{1}{2}r_0^2, \phi_0, z_0)}, \tag{7.101}$$

where ρ and ρ_0 are the corresponding densities.

Taking time rate of change of (7.100), we have

$$\frac{\sigma_0}{\sigma^2}\frac{\partial \sigma}{\partial \tau} + \frac{\partial(r\dot{r}, \phi, s)}{\partial(\frac{1}{2}r_0^2, \phi_0, s_0)} + \frac{\partial(\frac{1}{2}r^2, \dot{\phi}, s)}{\partial(\frac{1}{2}r_0^2, \phi_0, s_0)} + \frac{\partial(\frac{1}{2}r^2, \phi, \dot{s})}{\partial(\frac{1}{2}r_0^2, \phi_0, s_0)} = 0$$

which gives the Eulerian form of the mass continuity equation:

$$\frac{\partial \sigma}{\partial \tau} + \sigma\left(\frac{\partial(ru)}{r\partial r} + \frac{\partial v}{r\partial \phi} + \frac{\partial \dot{s}}{\partial s}\right) = 0, \tag{7.102}$$

where $\partial/\partial\tau = D/Dt$ is the time derivative following the motion of a fluid particle in the Lagrangian system.

The Lagrangian for the quasi-static primitive equations in a cylindrical coordinate system can be written as

$$L = \iiint \left[u\frac{\partial r}{\partial \tau} + (v + \tfrac{1}{2}fr)r\frac{\partial \phi}{\partial \tau} \right] \rho_0 r_0 dr_0 d\phi_0 dz_0 - H \tag{7.103}$$

where the Hamiltonian is

$$H = \iiint \left[\tfrac{1}{2}(u^2 + v^2) + E(\alpha, s) + gz \right] \rho_0 r_0 dr_0 d\phi_0 dz_0. \tag{7.104}$$

From (7.104) we see that H consists of kinetic, internal and potential energy. However, for the quasi-static primitive equations, the kinetic energy associated with the vertical component of velocity is neglected, which accounts for the fact that the zero occurs on the left hand side of (7.38) for the third component of Clebsch velocity representation. The internal energy E is associated with the thermodynamic processes of the system. Thus, E is a function of entropy s and the positions of the fluid parcel implicit in the specific volume α through (7.101). Here we expressed (7.103) and (7.104) in the z-coordinate for convenience of the discussions below. One can also write (7.103) and (7.104) in the s-coordinate, in which case H can be expressed as the Montgomery potential plus the kinetic energy with the mass-integral element changed to $\sigma_0 r_0 d\phi_0 ds_0$.

To prove that (7.103) and (7.104) form the correct variational principle for the quasi-static primitive equations, we take independent variations of this system.

δr:

$$\begin{aligned} &\int d\tau \iiint \left[u\frac{\partial(\delta r)}{\partial \tau} + (v + fr)\frac{\partial \phi}{\partial \tau}\delta r - \left(\frac{\partial E}{\partial \alpha}\right)_s \frac{1}{\rho_0}\frac{\partial(r\delta r, \phi, z)}{\partial(\frac{1}{2}r_0^2, \phi_0, z_0)} \right] \rho_0 d\Omega_0 \\ = &\int d\tau \iiint \left[-\frac{\partial u}{\partial \tau} + (v + fr)\frac{\partial \phi}{\partial \tau} - \frac{r}{\rho_0}\frac{\partial(p, \phi, z)}{\partial(\frac{1}{2}r_0^2, \phi_0, z_0)} \right] \delta r \rho_0 d\Omega_0 = 0, \end{aligned}$$

where $d\Omega_0 = r_0 dr_0 d\phi_0 dz_0$ is the volume-integral element. The last line is obtained through integration by parts and the use of Maxwell's equation of thermodynamics, $p = -(\partial E/\partial \alpha)_s$. Hence, the arbitrariness of δr implies that the integrand must vanish, giving

$$\frac{\partial u}{\partial \tau} - \left(f + \frac{v}{r}\right)v + \alpha r\frac{\partial(p, \phi, z)}{\partial(\frac{1}{2}r^2, \phi, z)} = 0, \tag{7.105}$$

where the last term in this expression can be rewritten as

$$\alpha r \frac{\partial(p,\phi,z)}{\partial(\frac{1}{2}r^2,\phi,z)} = \alpha r \frac{\partial(p,\phi,z)}{\partial(\frac{1}{2}r^2,\phi,s)} \frac{\partial(\frac{1}{2}r^2,\phi,s)}{\partial(\frac{1}{2}r^2,\phi,z)} = \alpha \left[\left(\frac{\partial p}{\partial r}\right)_s \frac{\partial z}{\partial s} - \left(\frac{\partial z}{\partial r}\right)_s \frac{\partial p}{\partial s} \right] \frac{\partial s}{\partial z}$$

$$= \alpha \left(\frac{\partial p}{\partial r}\right)_s + g \left(\frac{\partial z}{\partial r}\right)_s = \frac{\partial M}{\partial r}.$$

With this result, (7.105) becomes

$$\frac{\partial u}{\partial \tau} - \left(f + \frac{v}{r}\right) v + \frac{\partial M}{\partial r} = 0. \tag{7.106}$$

Similarly, the variation $\delta\phi$ of (7.103) and (7.104) gives

$$\frac{\partial v}{\partial \tau} + \left(f + \frac{v}{r}\right) u + \frac{\partial M}{r\partial \phi} = 0. \tag{7.107}$$

The variation in z component is also very straightforward:

δz:

$$\int d\tau \iiint \left[\left(\frac{\partial E}{\partial \alpha}\right)_s \frac{1}{\rho_0} \frac{\partial(\frac{1}{2}r^2,\phi,\delta z)}{\partial(\frac{1}{2}r_0^2,\phi_0,z_0)} + g\delta z \right] \rho_0 d\Omega_0$$

$$= \int d\tau \iiint \left[\alpha \frac{\partial(\frac{1}{2}r^2,\phi,p)}{\partial(\frac{1}{2}r^2,\phi,z)} + g\delta z \right] \rho_0 d\Omega_0 = 0.$$

This yields the quasi-static balanced relation

$$\alpha \frac{\partial p}{\partial z} + g = 0. \tag{7.108}$$

Then the proper form of (7.108) in the s-coordinate is

$$\frac{\partial M}{\partial s} = T. \tag{7.109}$$

In a similar fashion, we can show that variations δu and δv give the definitions of horizontal components of velocity: $u = \partial r/\partial \tau$ and $v = r\partial\phi/\partial\tau$.

The conservation principles are easily identified from (7.103) and (7.104) by their symmetry properties. In order to see conservation of energy, we note that $H = H(r,\phi,s,u,v,t)$, and then take the variations in its independent variables. to obtain

$$\int d\tau \iiint \left[u\frac{\partial(\delta r)}{\partial \tau} + \delta u \frac{\partial r}{\partial \tau} + (v + \tfrac{1}{2}fr)r\frac{\partial(\delta\phi)}{\partial \tau} + (v\delta r + r\delta v + fr\delta r)\frac{\partial \phi}{\partial \tau} \right.$$
$$\left. - \frac{\partial H^*}{\partial r}\delta r - \frac{\partial H^*}{\partial \phi}\delta\phi - \frac{\partial H^*}{\partial s}\delta s - \frac{\partial H^*}{\partial u}\delta u - \frac{\partial H^*}{\partial v}\delta v - \frac{\partial H^*}{\partial t}\delta t \right] \sigma_0 d\Omega_0 = 0,$$

where we have used the notation H^* to denote the integrand of H, which is the total energy per unit mass. Integrating by parts and grouping terms with the same variation, we get the following canonical equations:

$$\frac{\partial H^*}{\partial r} = fr + v - \dot{u}, \quad \frac{\partial H^*}{\partial \phi} = -r\dot{v} - v\dot{r} - fr\dot{r},$$

$$\frac{\partial H^*}{\partial s} = 0, \quad \frac{\partial H^*}{\partial u} = u, \quad \frac{\partial H^*}{\partial v} = v, \quad \frac{\partial H^*}{\partial t} = 0,$$

which, when substituted into the expression for total derivative of H^*, result in

$$\frac{dH^*}{dt} = 0. \tag{7.110}$$

This is the statement of energy conservation.

If the flow is axisymmetric, then the Hamiltonian is independent of ϕ. In this case, according to the second equation in (7.5), the corresponding momentum is an invariant, i.e., p_ϕ = const., which can be computed from (7.2). It gives

$$\tfrac{1}{2}fr^2 + rv = \text{constant}, \tag{7.111}$$

which is the statement of angular momentum conservation corresponding to (2.14) for axisymmetric flow.

The potential vorticity conservation is related to the particle relabeling symmetry (Salmon, 1983). Let us take the variations of (7.103) in the particle labeling coordinates (r_0, ϕ_0, s_0) while holding (r, ϕ, s) fixed. Noting the vertical velocity is absent in the Lagrangian for the quasi-static primitive system, we then have

$$\int d\tau \iiint \left[u\delta\left(\frac{\partial r}{\partial \tau}\right) + (rv + \tfrac{1}{2}fr^2)\delta\left(\frac{\partial \phi}{\partial \tau}\right)\right] \sigma_0 d\Omega_0 = 0, \tag{7.112}$$

where it is not difficult to show that

$$\delta\left(\frac{\partial r}{\partial \tau}\right) = -\frac{\partial r}{\partial r_0}\frac{\partial(\delta r_0)}{\partial \tau} - \frac{\partial r}{\partial \phi_0}\frac{\partial(\delta \phi_0)}{\partial \tau}, \tag{7.113}$$

$$\delta\left(\frac{\partial \phi}{\partial \tau}\right) = -\frac{\partial \phi}{\partial r_0}\frac{\partial(\delta r_0)}{\partial \tau} - \frac{\partial \phi}{\partial \phi_0}\frac{\partial(\delta \phi_0)}{\partial \tau}. \tag{7.114}$$

For adiabatic flow, $s = s_0$, so that (7.100) reduces to $\sigma_0/\sigma = \partial(\frac{1}{2}r^2, \phi)/\partial(\frac{1}{2}r_0^2, \phi_0)$. By taking the variation of this expression in the labeling coordinates, we obtain

$$r_0 \delta r_0 = -\frac{\partial \delta\psi}{\partial \phi_0}, \qquad r_0 \delta\phi_0 = \frac{\partial \delta\psi}{\partial r_0}. \tag{7.115}$$

Using (7.113)-(7.115), (7.112) becomes

$$\begin{aligned}
&-\int d\tau \iiint \left\{ \left[u\frac{\partial r}{\partial r_0} + (rv + \tfrac{1}{2}fr^2)\frac{\partial \phi}{\partial \phi_0} \right] \frac{\partial(\delta r_0)}{\partial \tau} + \left[u\frac{\partial r}{\partial \phi_0} \right.\right. \\
&\qquad\qquad \left.\left. +(v + \tfrac{1}{2}fr^2)\frac{\partial \phi}{\partial \phi_0} \right] \frac{\partial(\delta\phi_0)}{\partial \tau} \right\} \sigma_0 d\Omega_0 \\
&= \int d\tau \iiint \left\{ \delta r_0 \frac{\partial}{\partial \tau} \left[u\frac{\partial r}{\partial r_0} + (rv + \tfrac{1}{2}fr^2)\frac{\partial \phi}{\partial r_0} \right] + \delta\phi_0 \frac{\partial}{\partial \tau} \left[u\frac{\partial r}{\partial \phi_0} \right.\right. \\
&\qquad\qquad \left.\left. +(rv + \tfrac{1}{2}fr^2)\frac{\partial \phi}{\partial \phi_0} \right] \right\} \sigma_0 d\Omega_0 \\
&= \int d\tau \iiint \delta\psi \frac{\partial}{\partial \tau} \left[\frac{\partial(r, u)}{\partial(\frac{1}{2}r_0^2, \phi_0)} - \frac{\partial(rv + \frac{1}{2}fr^2, \phi)}{\partial(\frac{1}{2}r_0^2, \phi_0)} \right] \sigma_0 d\Omega_0 = 0,
\end{aligned}$$

which yields, due to the arbitrariness of $\delta\psi$,

$$\frac{\partial}{\partial \tau} \left[\left(f + \frac{\partial(rv)}{r\partial r} - \frac{\partial u}{r\partial \phi} \right) \frac{1}{\sigma} \right] = 0. \tag{7.116}$$

This is, of course, Ertel's theorem in the cylindrical coordinate system, which is equivalent to (2.29) for an adiabatic flow, as derived in Chapter 2.

We shall next consider variations in transformed phase space, the components of which are the Clebsch potentials R and Φ. Equations (7.36) and (7.37) are used as a set of constraints during the variations. We can rewrite the Lagrangian (7.103) in the form

$$\begin{aligned}
L = \iiint &\left[u\frac{\partial r}{\partial \tau} + (vr + \tfrac{1}{2}fr^2)\frac{\partial \phi}{\partial \tau} - \mathcal{M} + \frac{\partial \chi}{\partial t} + \tfrac{1}{4}f(R^2 - r^2)\frac{\partial \Phi}{\partial t} \right. \\
&\left. -\tfrac{1}{2}f(\Phi - \phi)R\frac{\partial R}{\partial t} \right] \sigma_0 d\Omega_0,
\end{aligned} \tag{7.117}$$

where $\mathcal{M}$ is defined in (7.42). Note that (7.117) is exactly the same as (7.103) when (7.103) is written in the s-coordinate.

The variations of (7.36) and (7.37) in R are

$$\delta u = \tfrac{1}{2}fR\frac{\partial \Phi}{\partial r}\delta R - \tfrac{1}{2}f(\Phi - \phi)\frac{\partial(\delta\frac{1}{2}R^2)}{\partial r}, \tag{7.118}$$

$$\delta(rv) = \tfrac{1}{2} f R \left(\frac{\partial \Phi}{\partial \phi} + 1 \right) \delta R - \tfrac{1}{2} f (\Phi - \phi) \frac{\partial (\delta \frac{1}{2} R^2)}{\partial \phi}. \tag{7.119}$$

Using these two relations and taking the variation of (7.117), we have

δR:

$$\int d\tau \iiint \left[\frac{\partial r}{\partial \tau} \delta u + r \frac{\partial \phi}{\partial \tau} \delta v + \tfrac{1}{2} f \frac{\partial \Phi}{\partial t} R \delta R - \tfrac{1}{2} f (\Phi - \phi) \frac{\partial \delta(\frac{1}{2} R^2)}{\partial t} \right.$$
$$\left. - \frac{\partial \mathcal{M}}{R \partial R} \delta(\tfrac{1}{2} R^2) \right] \sigma_0 d\Omega_0$$
$$= \int d\tau \iiint \left[\tfrac{1}{2} f \sigma_0 u \frac{\partial \Phi}{\partial r} + \frac{\partial}{r \partial r} \left(\tfrac{1}{2} f \sigma_0 r u (\Phi - \phi) \right) + \tfrac{1}{2} f \sigma_0 \frac{v}{r} \left(\frac{\partial \Phi}{\partial \phi} + 1 \right) \right.$$
$$+ \frac{\partial}{r \partial \phi} \left(\tfrac{1}{2} f \sigma_0 v (\Phi - \phi) \right) + \tfrac{1}{2} f \sigma_0 \frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial t} \left(\tfrac{1}{2} f \sigma_0 (\Phi - \phi) \right)$$
$$\left. - \frac{\partial \mathcal{M}}{R \partial R} \sigma_0 \right] \delta(\tfrac{1}{2} R^2) d\Omega_0 = 0,$$

which leads to

$$f R \frac{D \Phi}{D t} = \frac{\partial \mathcal{M}}{\partial R}, \tag{7.120}$$

by using the continuity equation.

Similarly, when we take the variation $\delta \Phi$ of (7.36) and (7.37) we get

$$\delta u = \tfrac{1}{4} f (R^2 - r^2) \frac{\partial (\delta \Phi)}{\partial r} - \tfrac{1}{2} f R \left(\frac{\partial R}{\partial r} + r \right) \delta \Phi, \tag{7.121}$$

$$\delta(rv) = \tfrac{1}{4} f (R^2 - r^2) \frac{\partial (\delta \Phi)}{\partial \phi} - \tfrac{1}{2} f R \frac{\partial R}{\partial \phi} \delta \Phi, \tag{7.122}$$

and therefore the variation of (7.117) can be calculated as

$\delta \Phi$:

$$\int d\tau \iiint \left[\frac{\partial r}{\partial \tau} \delta u + \frac{\partial \phi}{\partial \tau} \delta(rv) - \tfrac{1}{2} f R \frac{\partial R}{\partial t} \delta \Phi + \tfrac{1}{4} f (R^2 - r^2) \frac{\partial (\delta \Phi)}{\partial t} - \frac{\partial \mathcal{M}}{\partial \Phi} \delta \Phi \right] \sigma_0 d\Omega_0$$
$$= \int d\tau \iiint \left[-\tfrac{1}{2} f \sigma_0 u \left(R \frac{\partial R}{\partial r} + r \right) - \frac{\partial}{r \partial r} \left(\tfrac{1}{4} f \sigma_0 r u (R^2 - r^2) \right) - \tfrac{1}{2} f \sigma_0 v R \frac{\partial R}{r \partial \phi} \right.$$
$$- \frac{\partial}{r \partial \phi} \left(\tfrac{1}{4} f \sigma_0 v (R^2 - r^2) \right) - \tfrac{1}{2} f \sigma_0 R \frac{\partial R}{\partial t} - \frac{\partial}{\partial t} \left(\tfrac{1}{4} f \sigma_0 (R^2 - r^2) \right)$$
$$\left. - \frac{\partial \mathcal{M}}{\partial \Phi} \sigma_0 \right] \delta \Phi d\Omega_0 = 0,$$

which leads to

$$-f \frac{D R}{D t} = \frac{\partial \mathcal{M}}{R \partial \Phi}. \tag{7.123}$$

Again, equations (7.120) and (7.123) are the main results of this subsection, and they should be compared with the first two entries of (7.46) of the previous section.

7.2.3 The quasi-static primitive equations in spherical coordinates

We now extend the f-plane results obtained in the previous two subsections to the full spherical case, where the coordinates in Lagrangian system are

$$\lambda = \lambda(\lambda_0, \phi_0, s_0, \tau), \quad \phi = \phi(\lambda_0, \phi_0, s_0, \tau), \quad s = s(\lambda_0, \phi_0, s_0, \tau).$$

The mass continuity can be expressed as

$$d(\text{mass}) = \sigma a^2 \cos\phi d\lambda d\phi ds = \sigma_0 a^2 \cos\phi_0 d\lambda_0 d\phi_0 ds_0, \tag{7.124}$$

which can be written in the Jacobian form

$$\frac{\sigma_0}{\sigma} = \frac{\partial(\lambda, \sin\phi, s)}{\partial(\lambda_0, \sin\phi_0, s_0)}, \tag{7.125}$$

where σ and σ_0 are the pseudodensity at time τ and at the initial time, respectively. In the z-coordinate system, the proper form of (7.125) is

$$\frac{\rho_0}{\rho} = \frac{\partial(\lambda, \sin\phi, z)}{\partial(\lambda_0, \sin\phi_0, z_0)}, \tag{7.126}$$

where ρ and ρ_0 are the corresponding densities.

Taking time rate of change of (7.125), we have

$$\frac{\sigma_0}{\sigma^2}\frac{\partial\sigma}{\partial\tau} + \frac{\partial(\dot\lambda, \sin\phi, s)}{\partial(\lambda_0, \sin\phi_0, s_0)} + \frac{\partial(\lambda, \dot\phi\cos\phi, s)}{\partial(\lambda_0, \sin\phi_0, s_0)} + \frac{\partial(\lambda, \sin\phi, \dot s)}{\partial(\lambda_0, \sin\phi_0, s_0)} = 0,$$

which gives the Eulerian form of the mass continuity equation:

$$\frac{\partial\sigma}{\partial\tau} + \sigma\left(\frac{\partial u}{a\cos\phi\partial\lambda} + \frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial\dot s}{\partial s}\right) = 0, \tag{7.127}$$

where $\partial/\partial\tau = D/Dt$ is the time derivative following the motion of a fluid particle in the Lagrangian system.

The Lagrangian for the quasi-static primitive equations in a spherical coordinate system can be written as

$$L = \iiint \left[a\cos\phi(u + \Omega a\cos\phi)\frac{\partial\lambda}{\partial\tau} + va\frac{\partial\phi}{\partial\tau}\right]\rho_0 a^2\cos\phi_0 d\lambda_0 d\phi_0 dz_0 - H \tag{7.128}$$

where the Hamiltonian is

$$H = \iiint \left[\tfrac{1}{2}(u^2 + v^2) + E(\alpha, s) + gz\right]\rho_0 a^2 \cos\phi_0 d\lambda_0 d\phi_0 dz_0. \tag{7.129}$$

Note that H consists of kinetic, internal and potential energy, and that the kinetic energy associated with vertical velocity is omitted. This approximation is responsible for the fact that a zero appears on the left hand side of (7.63) for the third component of the Clebsch representation. The internal energy E is associated with the thermodynamic processes of the system. Thus, E is a function of entropy s and the positions of the fluid parcel implicit in the specific volume α through (7.126). Here (7.128) and (7.129) are formulated in z-coordinate. They can also be written in s-coordinate in which case H is expressed as the Montgomery potential plus the kinetic energy with the mass integral-element changed to $\sigma_0 a^2 \cos\phi_0 d\lambda_0 d\phi_0 ds_0$. Both formulations will be used in the following discussions for convenience.

To prove that (7.128) and (7.129) form the correct variational principle for the quasi-static primitive equations on the sphere, we take independent variations of this system.

$\delta\lambda$:

$$\int d\tau \iiint \left[a\cos\phi(u+\Omega a\cos\phi)\frac{\partial(\delta\lambda)}{\partial\tau} - \left(\frac{\partial E}{\partial\alpha}\right)_s \frac{1}{\rho_0}\frac{\partial(\delta\lambda,\sin\phi,z)}{\partial(\lambda_0,\sin\phi_0,z_0)}\right]\rho_0 d\Omega_0$$
$$= \int d\tau \iiint \left[\frac{\partial}{\partial\tau}\left(a\cos\phi(u+\Omega a\cos\phi)\right) + \frac{1}{\rho_0}\frac{\partial(p,\sin\phi,z)}{\partial(\lambda_0,\sin\phi_0,z_0)}\right]\delta\lambda\rho_0 d\Omega_0 = 0,$$

where $d\Omega_0 = a^2\cos\phi_0 d\lambda_0 d\phi_0 ds_0$ is the volume-integral element. The last line is obtained through integration by parts and using Maxwell's equation of thermodynamics, $p = -(\partial E/\partial\alpha)_s$. Hence, the arbitrariness of $\delta\lambda$ implies that the integrand must vanish, giving

$$\frac{\partial}{\partial\tau}\left[a\cos\phi(u+\Omega a\cos\phi)\right] + \alpha\frac{\partial(p,\sin\phi,z)}{\partial(\lambda,\sin\phi,z)} = 0, \tag{7.130}$$

where the last term in this expression can be rewritten as

$$\alpha\frac{\partial(p,\sin\phi,z)}{\partial(\lambda,\sin\phi,z)} = \alpha\frac{\partial(p,\sin\phi,z)}{\partial(\lambda,\sin\phi,s)}\frac{\partial(\lambda,\sin\phi,s)}{\partial(\lambda,\sin\phi,z)} = \alpha\left[\left(\frac{\partial p}{\partial\lambda}\right)_s\frac{\partial z}{\partial s} - \left(\frac{\partial z}{\partial\lambda}\right)_s\frac{\partial p}{\partial s}\right]\frac{\partial s}{\partial z}$$
$$= \alpha\left(\frac{\partial p}{\partial\lambda}\right)_s + g\left(\frac{\partial z}{\partial\lambda}\right)_s = \frac{\partial M}{\partial\lambda}.$$

With this result, (7.130) becomes

$$\frac{\partial u}{\partial\tau} - \left(2\Omega\sin\phi + \frac{u\tan\phi}{a}\right)v + \frac{\partial M}{a\cos\phi\partial\lambda} = 0. \tag{7.131}$$

Similarly, the variation $\delta\phi$ of (7.128) and (7.129) gives

$$\frac{\partial v}{\partial \tau} + \left(2\Omega \sin\phi + \frac{u \tan\phi}{a}\right) u + \frac{\partial M}{a\partial\phi} = 0. \tag{7.132}$$

The variation in z component is

δz:

$$\int d\tau \iiint \left[\left(\frac{\partial E}{\partial \alpha}\right)_s \frac{1}{\rho_0} \frac{\partial(\lambda, \sin\phi, \delta z)}{\partial(\lambda_0, \sin\phi_0, z_0)} + g\delta z\right] \rho_0 d\Omega_0$$

$$= \int d\tau \iiint \left[\alpha \frac{\partial(\lambda, \sin\phi, p)}{\partial(\lambda, \sin\phi, z)} + g\delta z\right] \rho_0 d\Omega_0 = 0,$$

which yields the quasi-static relation

$$\alpha \frac{\partial p}{\partial z} + g = 0. \tag{7.133}$$

If (7.133) is written in s-coordinate, it takes the form

$$\frac{\partial M}{\partial s} = T. \tag{7.134}$$

The variations δu and δv give the definitions of horizontal components of velocity: $u = a\cos\phi\partial\lambda/\partial\tau$ and $v = a\partial\phi/\partial\tau$. Now collecting (7.127), (7.131), (7.132) and (7.134), we have derived the complete set of quasi-static primitive equations on the sphere from the Lagrangian (7.128).

The conservation principles are easily identified from (7.128) and (7.129) through the symmetry properties of the Hamiltonian. In order to see the conservation of energy, we note that $H = H(\lambda, \phi, s, u, v, t)$, and then take the variations in its independent variables to obtain

$$\begin{aligned}\int d\tau \iiint \Big[& a\cos\phi(u + \Omega a\cos\phi)\frac{\partial(\delta\lambda)}{\partial\tau} - (u + 2\Omega a\cos\phi) a \sin\phi \frac{\partial\lambda}{\partial\tau}\delta\phi \\ & + va\frac{\partial(\delta\phi)}{\partial\tau} + a\cos\phi\frac{\partial\lambda}{\partial\tau}\delta u + a\frac{\partial\phi}{\partial\tau}\delta v - \frac{\partial H^*}{\partial\lambda}\delta\lambda - \frac{\partial H^*}{\partial\phi}\delta\phi \\ & - \frac{\partial H^*}{\partial s}\delta s - \frac{\partial H^*}{\partial u}\delta u - \frac{\partial H^*}{\partial v}\delta v - \frac{\partial H^*}{\partial t}\delta t\Big] \sigma_0 d\Omega_0 = 0,\end{aligned}$$

where H^* is the integrand of H, which stands for the total energy per unit mass. Integrating by parts and grouping terms with the same variation, we get the following canonical equations:

$$\frac{\partial H^*}{\partial\lambda} = -\frac{\partial}{\partial\tau}\left[a\cos\phi(u + \Omega a\cos\phi)\right],$$

$$\frac{\partial H^*}{\partial \phi} = -a\dot{v} - (u + 2\Omega a \cos\phi) a \sin\phi \frac{\partial \lambda}{\partial \tau},$$

$$\frac{\partial H^*}{\partial s} = 0, \quad \frac{\partial H^*}{\partial u} = u, \quad \frac{\partial H^*}{\partial v} = v, \quad \frac{\partial H^*}{\partial t} = 0,$$

which, when substituted into the expression for total derivative of H^*, result in

$$\frac{dH^*}{dt} = 0. \tag{7.135}$$

This is the statement of energy conservation.

If the flow is zonally symmetric, then the Hamiltonian is independent of λ. In this case, according to the second equation in (7.5), the corresponding momentum is an invariant, i.e., p_λ = const., which can be computed from (7.2). It gives

$$a \cos\phi (u + \Omega a \cos\phi) = \text{constant}, \tag{7.136}$$

which is the statement of angular momentum conservation corresponding to (3.41) for zonally symmetric flow.

The potential vorticity conservation is related to the particle relabeling symmetry (Salmon, 1983). Let us take the variations of (7.128) in the particle labeling coordinates (λ_0, ϕ_0, s_0) while holding (r, ϕ, s) fixed. Noting the vertical velocity is absent in the Lagrangian for the quasi-static primitive system, we have

$$\int d\tau \iiint \left[a \cos\phi (u + \Omega a \cos\phi) \delta \left(\frac{\partial \lambda}{\partial \tau} \right) + va\delta \left(\frac{\partial \phi}{\partial \tau} \right) \right] \sigma_0 d\Omega_0 = 0. \tag{7.137}$$

It is not difficult to show that

$$\delta \left(\frac{\partial \lambda}{\partial \tau} \right) = -\frac{\partial \lambda}{\partial \lambda_0} \frac{\partial (\delta \lambda_0)}{\partial \tau} - \frac{\partial \lambda}{\partial \phi_0} \frac{\partial (\delta \phi_0)}{\partial \tau}, \tag{7.138}$$

$$\delta \left(\frac{\partial \phi}{\partial \tau} \right) = -\frac{\partial \phi}{\partial \lambda_0} \frac{\partial (\delta \lambda_0)}{\partial \tau} - \frac{\partial \phi}{\partial \phi_0} \frac{\partial (\delta \phi_0)}{\partial \tau}. \tag{7.139}$$

For adiabatic flow, $s = s_0$, so that (7.125) reduces to $\sigma_0/\sigma = \partial(\lambda, \sin\phi)/\partial(\lambda_0, \sin\phi_0)$. By taking the variation of this expression in the labeling coordinates, we obtain

$$\cos\phi_0 \delta\lambda_0 = -\frac{\partial \delta\psi}{\partial \phi_0}, \quad \cos\phi_0 \delta\phi_0 = \frac{\partial \delta\psi}{\partial \lambda_0}. \tag{7.140}$$

Substituting in (7.138)–(7.140), (7.137) becomes

$$\int d\tau \iiint \left\{ \delta\lambda_0 \frac{\partial}{\partial\tau}\left[a\cos\phi(u+\Omega a\cos\phi)\frac{\partial\lambda}{\partial\lambda_0}+va\frac{\partial\phi}{\partial\lambda_0}\right]+\delta\phi_0\frac{\partial}{\partial\tau}\left[a\cos\phi(u\right.\right.$$
$$\left.\left.+\Omega a\cos\phi)\frac{\partial\lambda}{\partial\phi_0}+va\frac{\partial\phi}{\partial\phi_0}\right]\right\}\sigma_0 d\Omega_0$$
$$=\int d\tau \iiint \delta\psi\frac{\partial}{\partial\tau}\left[\frac{\partial(\lambda,\cos\phi(u+\Omega a\cos\phi))}{a\partial(\lambda_0,\sin\phi_0)}-\frac{\partial(v,\phi)}{a\partial(\lambda_0,\sin\phi_0)}\right]\sigma_0 d\Omega_0=0,$$

which yields, due to the arbitrariness of $\delta\psi$,

$$\frac{\partial}{\partial\tau}\left[\left(2\Omega\sin\phi+\frac{\partial v}{a\cos\phi\partial\lambda}-\frac{\partial(u\cos\phi)}{a\cos\phi\partial\phi}\right)\frac{1}{\sigma}\right]=0. \tag{7.141}$$

This is, of course, Rossby-Ertel's theorem in the spherical coordinate system, which is equivalent to (3.27) for an adiabatic flow derived in Chapter 3.

We shall next consider variations in transformed space, two components of which are the Clebsch potentials Λ and Φ. Equations (7.61) and (7.62) are used as a set of constriants during the variations. We can rewrite the Lagrangian (7.128) in the form

$$L=\iiint\left[a\cos\phi(u+\Omega a\cos\phi)\frac{\partial\lambda}{\partial\tau}+va\frac{\partial\phi}{\partial\tau}-\mathcal{M}+\frac{\partial\chi}{\partial t}-\tfrac{1}{2}\Omega a^2(\sin^2\Phi-\sin^2\phi)\frac{\partial\Lambda}{\partial t}\right.$$
$$\left.\tfrac{1}{2}\Omega a^2(\Lambda-\lambda)\frac{\partial(\sin^2\Phi)}{\partial t}\right]\sigma_0 d\Omega_0 \tag{7.142}$$

where $\mathcal{M}$ is defined in (7.67).

The variations δR of (7.61) and (7.62) are

$$\delta u=-\tfrac{1}{2}\Omega a^2(\sin^2\Phi-\sin^2\phi)\frac{\partial(\delta\Lambda)}{a\cos\phi\partial\lambda}+\tfrac{1}{2}\Omega a^2\frac{\partial(\sin^2\Phi)}{a\cos\phi\partial\lambda}\delta\Lambda, \tag{7.143}$$

$$\delta v=-\tfrac{1}{2}\Omega a^2(\sin^2\Phi-\sin^2\phi)\frac{\partial(\delta\Lambda)}{a\partial\phi}+\tfrac{1}{2}\Omega a^2\frac{\partial(\sin^2\Phi+\sin^2\phi)}{a\partial\phi}\delta\Lambda. \tag{7.144}$$

Using these two relations and taking the variation of (7.142), we have

δR:

$$\int d\tau\iiint\left[a\cos\phi\frac{\partial\lambda}{\partial\tau}\delta u+a\frac{\partial\phi}{\partial\tau}\delta v-\tfrac{1}{2}\Omega a^2(\sin^2\Phi-\sin^2\phi)\frac{\partial(\delta\Lambda)}{\partial t}\right.$$
$$\left.+\tfrac{1}{2}\Omega a^2\frac{\partial\delta(\sin^2\Phi)}{\partial t}\delta\Lambda-\frac{\partial\mathcal{M}}{\partial\Lambda}\delta\Lambda\right]\sigma_0 d\Omega_0$$
$$=\int d\tau\iiint\left[\tfrac{1}{2}\Omega a^2\sigma_0 u\frac{\partial(\sin^2\Phi)}{a\cos\phi\partial\lambda}+\frac{\partial}{a\cos\phi\partial\lambda}\left(\tfrac{1}{2}\Omega a^2\sigma_0 u(\sin^2\Phi-\sin^2\phi)\right)\right.$$

$$+\tfrac{1}{2}\Omega a^2\sigma_0 v\frac{\partial(\sin^2\Phi+\sin^2\phi)}{a\partial\phi}+\frac{\partial}{a\cos\phi\partial\phi}\left(\tfrac{1}{2}\Omega a^2\sigma_0 v\cos\phi(\sin^2\Phi-\sin^2\phi)\right)$$
$$+\tfrac{1}{2}\Omega a^2\sigma_0\frac{\partial(\sin^2\Phi)}{\partial t}+\frac{\partial}{\partial t}\left(\tfrac{1}{2}\Omega a^2\sigma_0(\sin^2\Phi-\sin^2\phi)\right)$$
$$\left.-\frac{\partial\mathcal{M}}{\partial\Lambda}\sigma_0\right]\delta\Lambda d\Omega_0=0,$$

which leads to

$$2\Omega\sin\Phi a\frac{D\Phi}{Dt}=\frac{\partial\mathcal{M}}{a\cos\Phi\partial\Lambda}, \tag{7.145}$$

by using the continuity equation.

Similarly, when we take the variation $\delta\Phi$ of (7.61) and (7.62) we get

$$\delta u=-\tfrac{1}{2}\Omega a^2\frac{\partial(\Lambda+\lambda)}{a\cos\phi\partial\lambda}\delta(\sin^2\Phi)+\tfrac{1}{2}\Omega a^2(\Lambda-\lambda)\frac{\partial\delta(\sin^2\Phi)}{a\cos\phi\partial\lambda}, \tag{7.146}$$

$$\delta v=-\tfrac{1}{2}\Omega a^2\frac{\partial\Lambda}{a\partial\phi}\delta(\sin^2\Phi)+\tfrac{1}{2}\Omega a^2(\Lambda-\lambda)\frac{\partial\delta(\sin^2\Phi)}{a\partial\phi}, \tag{7.147}$$

and therefore the variation of (7.142) can be calculated as

$\delta\Phi$:

$$\int d\tau\iiint\left[a\cos\phi\frac{\partial\lambda}{\partial\tau}\delta u+a\frac{\partial\phi}{\partial\tau}\delta v-\tfrac{1}{2}\Omega a^2\frac{\partial\Lambda}{\partial t}\delta(\sin^2\Phi)+\tfrac{1}{2}\Omega a^2(\Lambda-\lambda)\frac{\partial\delta(\sin^2\Phi)}{\partial t}\right.$$
$$\left.-\frac{\partial\mathcal{M}}{\partial(\sin^2\Phi)}\delta(\sin^2\Phi)\right]\sigma_0 d\Omega_0$$

$$=\int d\tau\iiint\left[-\tfrac{1}{2}\Omega a^2\sigma_0 u\frac{\partial(\Lambda+\lambda)}{a\cos\phi\partial\lambda}-\frac{\partial}{a\cos\phi\partial\lambda}\left(\tfrac{1}{2}\Omega a^2\sigma_0 u(\Lambda-\lambda)\right)-\tfrac{1}{2}\Omega a^2\sigma_0 v\frac{\partial\Lambda}{a\partial\phi}\right.$$
$$-\frac{\partial}{a\cos\phi\partial\phi}\left(\tfrac{1}{2}\Omega a^2\sigma_0 v\cos\phi(\Lambda-\lambda)\right)-\tfrac{1}{2}\Omega a^2\sigma_0\frac{\partial\Lambda}{\partial t}-\frac{\partial}{\partial t}\left(\tfrac{1}{2}\Omega a^2\sigma_0(\Lambda-\lambda)\right)$$
$$\left.-\frac{\partial\mathcal{M}}{\partial(\sin^2\Phi)}\sigma_0\right]\delta(\sin^2\Phi)d\Omega_0=0,$$

which leads to

$$-2\Omega\sin\Phi a\cos\Phi\frac{D\Lambda}{Dt}=\frac{\partial\mathcal{M}}{a\partial\Phi}. \tag{7.148}$$

Again, equations (7.145) and (7.148) are the main results for this subsection, and they should be compared with the first two entries of (7.71) of the previous section.

7.3 Balanced dynamics from a simplified Hamilton's principle and Clebsch potentials

In some balanced dynamical models such as the semigeostrophic theory (Hoskins, 1975; Hoskins and Draghici, 1977; Schubert et al., 1989; Magnusdottir and Schubert, 1990, 1991), the symmetric balanced theories (Schubert and Hack, 1983; Schubert and Alworth, 1987; Hack *et al.*, 1989; Schubert *et al.*, 1991) and the mixed balanced theory (the current work, Chapters 2 and 3), the geostrophic coordinates, the potential radius and potential latitude are introduced primarily as mathematical devices. Utilization of these mathematical tools leads to simplified dynamics which are formally similar to the quasi-geostrophic dynamics. In section 7.1 we have shown that these devices may be regarded as simplified Clebsch velocity decompositions. While the general form of the Clebsch representation can be used to transform the full variational principle to its potential phase space, from which the canonical momentum equations corresponding to the primitive equations are derivable, one may question whether such a methodology is carried through in balanced dynamics. In this section, we identify the variational principles for several balanced models, especially for the models developed in Chapters 2 and 3, by making approximations in the full Lagrangian, and then transform these variational principles to their potential phase space by using simplified Clebsch velocity potentials. The variations of these transformed Hamilton principles lead to the canonical momentum equations for different balanced models.

7.3.1 The Hamilton principle and Clebsch transformation associated with the semigeostrophic shallow water equations

When we partition the total flow field into geostrophic and ageostrophic flows, i.e.,

$$u = u_g + \epsilon u_a, \quad v = v_g + \epsilon v_a, \tag{7.149}$$

we may assume that the deviation of the flow field from geostrophy is small. This smallness is denoted by a small parameter ϵ in (7.149). It is arguable, however, that this assumption may not be valid in situations where strong convection and Ekman layers are considered. We note that in the formal development of balanced theories, especially those derived through scale analysis (e.g., McWilliams and Gent, 1980), much weaker conditions are needed.

Under assumption (7.149), the Lagrangian (7.81) can be written

$$L = \iint \left[(u_g - \Omega y)\frac{\partial x}{\partial \tau} + (v_g + \Omega x)\frac{\partial y}{\partial \tau} \right] h_0 dx_0 dy_0 - H + O(\epsilon), \tag{7.150}$$

where

$$H = \iint \tfrac{1}{2}(u_g^2 + v_g^2 + gh) h_0 dx_0 dy_0 \tag{7.151}$$

is the modified Hamiltonian. Equation (7.150) without the $O(\epsilon)$ term was considered by Salmon (1985), who referred to it as L_1 dynamics.

Since u_g and v_g are geostrophic winds, they are functions of particle locations even under the Lagrangian description. Coupled with (x, y), they form a distorted phase space whose projection is the true phase space. We now take the variations of (7.150) in (x, y). For δx:

$$\begin{aligned} \int d\tau \iint \Big[\frac{\partial x}{\partial \tau}\delta u_g + (u_g - \Omega y)\frac{\partial(\delta x)}{\partial \tau} + (\delta v_g + \Omega \delta x)\frac{\partial y}{\partial \tau} - u_g \delta u_g - v_g \delta v_g \\ - \tfrac{1}{2} g \delta h \Big] h_0 dS_0 + O(\epsilon) \\ = \int d\tau \iint \Big[-\delta x \frac{\partial}{\partial \tau}(u_g - \Omega y) + \Omega \frac{\partial y}{\partial \tau}\delta x - \tfrac{1}{2}\rho g \frac{\partial(h^2, y)}{\partial(x_0, y_0)}\delta x + \epsilon u_a \delta u_g \\ + \epsilon v_a \delta v_g \Big] h_0 dS_0 + O(\epsilon) = 0, \end{aligned}$$

where $dS_0 = dx_0 dy_0$ is the area-integral element. Because we have already made $O(\epsilon)$ approximations in the Lagrangian, we are not afraid to put the last two terms inside the brackets, in position to be neglected since they are also $O(\epsilon)$ terms. Thus, the variation δx of the modified Hamilton principle yields

$$\frac{\partial u_g}{\partial \tau} - fv + g\frac{\partial h}{\partial x} = 0, \tag{7.152}$$

where $f = 2\Omega$. Similarly, the variation in y gives

$$\frac{\partial v_g}{\partial \tau} + fu + g\frac{\partial h}{\partial y} = 0. \tag{7.153}$$

The foregoing approximation in the Lagrangian does not affect the mass continuity arguement so that we still have the mass continuity equation expressed in the Jacobian form (7.79) whose total time derivative will give

$$\frac{\partial h}{\partial \tau} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{7.154}$$

It is obvious that we have derived the set of GM equations (7.152)–(7.154) from an approximated Hamilton principle. Since the particle relabeling symmetry and the time translation property in the Hamiltonian are preserved, the potential vorticity and energy invariants for such an approximate system are guaranteed by Noether's theorem. One can prove these simply by going through exactly the same procedures as we did in the previous section for the primitive system.

Our next task is to show that the reduced form of Clebsch representation of velocity, i.e., the geostrophic coordinates, coupled with the simplified Hamilton principle will result in a variational principle in transformed space whose variations with respect to transformed coordinates will lead to the canonical momentum equations. These canonical equations form the basis of semigeostrophic theory.

We denote the set of reduced Clebsch velocity potentials associated with the vorticity coordinates as

$$X = x + F, \quad Y = y + G. \tag{7.155}$$

When $F = v_g/f$ and $G = -u_g/f$, (7.155) expresses the exact geostrophic coordinates.

Following Salmon (1985), the transformation of (7.150) from (x, y) space to (X, Y) space is done inversely, i.e.,

$$\begin{aligned}
&\iint \left[-\tfrac{1}{2}fY\delta X + \tfrac{1}{2}fX\delta Y\right] h_0 dS_0 \\
= &\iint \left[-\tfrac{1}{2}f(y+G)\delta(x+F) + \tfrac{1}{2}f(x+F)\delta(y+G)\right] h_0 dS_0 \\
= &\iint \Big[-\tfrac{1}{2}f(y+G)\delta x - \delta(\tfrac{1}{2}fyF) + \tfrac{1}{2}fF\delta y + \tfrac{1}{2}f(x+F)\delta y - \delta(\tfrac{1}{2}fxG) \\
&\qquad -\tfrac{1}{2}fG\delta x\Big] h_0 dS_0 + O(R_o^2 fL^2) \\
= &\iint \left[-\tfrac{1}{2}f(y+2G)\delta x + \tfrac{1}{2}f(x+F)\delta y\right] h_0 dS_0 + O(R_o^2 fL^2) \\
= &\iint \left[(u_g - \tfrac{1}{2}fy)\delta x + (v_g + \tfrac{1}{2}fx)\delta y\right] h_0 dS_0 + O(R_o^2 fL^2),
\end{aligned}$$

where we need to make the following remarks in order for this transformation to be carried through: (1) the exact differentiation terms $\delta(\,\cdot\,)$ vanish due to the stationary property of endpoints when full variation is concerned; (2) here $O(R_o^2 f L^2)$ stands for quadratic terms in F and G, where R_o is the Rossby number and L is the characteristic length scale. The last line in this expression yields L_1 dynamics plus this quadratic term. Therefore we can write the transformed Hamilton principle as

$$\delta \int d\tau \left[\iint \left(-\tfrac{1}{2} f Y \frac{\partial X}{\partial \tau} + \tfrac{1}{2} f X \frac{\partial Y}{\partial \tau} \right) h_0 dS_0 - H \right] = 0, \tag{7.156}$$

where H is defined in (7.151).

We now take variations of (7.156) in transformed space, which yields

$$\delta X: \qquad f\frac{DY}{Dt} = \frac{\partial H}{\partial X},$$

$$\delta Y: \qquad -f\frac{DX}{Dt} = \frac{\partial H}{\partial Y}.$$

Salmon (1985) has proved that the functional derivatives $\partial H/\partial X = g\partial\mathcal{H}/\partial X$ and $\partial H/\partial Y = g\partial\mathcal{H}/\partial Y$ where $g\mathcal{H} = gh + \frac{1}{2}(u_g^2 + v_g^2)$. Therefore, the variations lead to

$$f\frac{DY}{Dt} = g\frac{\partial \mathcal{H}}{\partial X}, \tag{7.157}$$

$$-f\frac{DX}{Dt} = g\frac{\partial \mathcal{H}}{\partial Y}. \tag{7.158}$$

Equations (7.157) and (7.158) are the canonical forms of momentum equations (7.152) and (7.153). These canonical equations are similar to what we have obtained in the previous sections, e.g., (7.21), (7.95) and (7.98), with the essential difference in the definition of $\mathcal{H}$.

To complete this balanced model, one need only note that the vorticity equation can be derived from (7.157) and (7.158), which takes the same form as (7.24) derived in section 7.1. Since the mass continuity equation is unaltered, the potential vorticity is followed by combining the vorticity equation and the mass continuity equation. With the definition (7.26), one can rewrite the potential vorticity equation as the potential height equation. These equations all take the same form as discussed in section 7.1. The difference here is that now (7.26) is invertable since the velocity field and mass field are interrelated by the geostrophic assumption, and they can all be expressed in terms of a single variable $\mathcal{H}$. Substitution of $\mathcal{H}$ into (7.23) and then (7.27) gives a complete prediction cycle.

7.3.2 The Hamilton principle and Clebsch transformation associated with the mixed-balance equations on an f-plane

Let us now consider a more complicated physical situation. The flow now is three dimensional and possesses strong curvature. Assuming the curvature of this flow can be depicted by its tangential component of velocity, we shall then partition this tangential velocity into gradient balanced flow and the deviation from this balance. At the same time we maintain the partition of the radial wind into geostrophic and ageostrophic components, but the geostrophic part is modified by a curvature factor. In this case, (7.149) becomes

$$u = \gamma u_g + \epsilon u_a, \quad v = v_g + \epsilon v_a, \tag{7.158}$$

where u_g is the geostrophic momentum defined in (2.42), v_g is the gradient wind defined in (2.43), and γ is a curvature parameter given in (2.44). Again we assume that the momentum components that deviate from the geostrophic and gradient values are small.

Under this approximation and using the s-coordinate in the vertical, the Lagrangian (7.103) is modified to

$$L = \iiint \left[\gamma u_g \frac{\partial r}{\partial \tau} + (v_g + \tfrac{1}{2} f r) r \frac{\partial \phi}{\partial \tau} - M^*\right] \sigma_0 r_0 dr_0 d\phi_0 ds_0 + O(\epsilon), \tag{7.159}$$

where

$$M^* = M + \tfrac{1}{2}(u_g^2 + v_g^2), \tag{7.160}$$

which represents the total energy in the s-coordinate. M is the Montgomery potential $M = c_p T + gz$, a quantity that combines the internal and potential energy.

We next take variations of (7.159) in physical coordinates (r, ϕ, z).

δr:

$$\int d\tau \iiint \left[\gamma u_g \frac{\partial(\delta r)}{\partial \tau} + \frac{\partial r}{\partial \tau}\delta(\gamma u_g) + (v_g + fr)\frac{\partial \phi}{\partial \tau}\delta r + r\frac{\partial \phi}{\partial \tau}\delta v_g \right.$$
$$\left. - \frac{\partial M}{\partial r}\delta r - u_g \delta u_g - v_g \delta v_g\right] \sigma_0 d\Omega_0 + O(\epsilon)$$

$$= \int d\tau \iiint \left[-\frac{\partial(\gamma u_g)}{\partial \tau} + (v_g + fr)\frac{\partial \phi}{\partial \tau} - \frac{\partial M}{\partial r}\right] \delta r \sigma_0 d\Omega_0 + O(\epsilon) = 0,$$

where it should be noted that the difference between the second and sixth terms, the fourth and seventh terms result in $O(\epsilon)$ terms, which can all be put in the small residual. The notation $d\Omega_0 = r_0 dr_0 d\phi_0 ds_0$ stands for the volume-integral element.

Assuming that the change of the curvature distortion factor following a particle motion can be neglected, we then obtain from the arbitrariness of δr in this variation

$$\frac{\partial u_g}{\partial \tau} - \left(f + \frac{v_g}{r}\right)\frac{v}{\gamma} + \frac{\partial M}{\gamma \partial r} = 0. \tag{7.161}$$

Similarly, for the variation $\delta\phi$ of (7.159)

$\delta\phi$:

$$\int d\tau \iiint \left[\frac{\partial r}{\partial \tau}\delta(\gamma u_g) + (v_g + \tfrac{1}{2} f r) r \frac{\partial(\delta\phi)}{\partial \tau} + r\frac{\partial \phi}{\partial \tau}\delta v_g - \frac{\partial M}{\partial \phi}\delta\phi \right.$$
$$\left. - u_g \delta u_g - v_g \delta v_g\right] \sigma_0 d\Omega_0 + O(\epsilon)$$
$$= \int d\tau \iiint \left[-\frac{\partial}{\partial \tau}(r v_g + \tfrac{1}{2} f r^2 - \frac{\partial M}{\partial \phi}\right] \delta r \sigma_0 d\Omega_0 + O(\epsilon) = 0,$$

which gives

$$\frac{\partial v_g}{\partial \tau} + \left(f + \frac{v_g}{r}\right) u + \frac{\partial M}{r \partial \phi} = 0. \tag{7.162}$$

Since the approximation made in the Lagrangian does not change the mass continuity equation (7.100), and the internal and potential energy are left unaltered, one can easily show that the variation of (7.159) in z and the time derivative with repect to (7.100) result in the hydrostatic equation

$$\frac{\partial M}{\partial s} = T, \tag{7.163}$$

and the mass continuity equation

$$\frac{\partial \sigma}{\partial \tau} + \sigma\left(\frac{\partial (r u)}{r \partial r} + \frac{r \partial \phi}{\partial y} + \frac{\partial \dot{s}}{\partial s}\right) = 0, \tag{7.164}$$

respectively. Collecting (7.161)–(7.164), one can see that this set of equations is exactly the set we derived in Chapter 2 with the combined geostrophic and gradient momentum approximation. The fact that this approximated system retains the particle labeling symmetry, the cyclic and time translation conditions in the Hamiltonian explains why we could derive all the corresponding conservation laws in Chapter 2.

Craig (1991) claimed to have developed a three dimensional theory for a balanced vortex from Hamilton's principle. Using the scale analysis with the assumption that the radial wind is much smaller than the tangential wind, he was able to formulate an approximate Lagrangian in which the radial wind is completely absent. Variations of his modified Lagrangian in physical space result in a set of dynamical equations which are nearly the same as Eliassen's axisymmetric balanced vortex equations. However, in transformed space he was surprisingly able to get the canonical momentum equations in three dimensions. The L_1 dynamics that we obtained, i.e., (7.159), is quite different from Craig's in that the radial wind, though small, is still retraceable in the approximate Lagrangian so that variations of such a Lagrangian in physical space produce the particle accelerations in the radial direction [ref. (7.161)]. It is this term that captures the physics needed to alter the axisymmetric balanced flow. In Craig's theory, however, such asymmetric mechanisms are not included. We next prove that our approximate system, when transformed to potential radius and geostrophic azimuth space, will yield three dimensional canonical momentum equations identical to those of Craig (1991).

To transform this dynamical system, let us denote the set of reduced Clebsch velocity potentials associated with the vorticity coordinates as

$$\tfrac{1}{2}R^2 = \tfrac{1}{2}r^2 + F, \quad \Phi = \phi + G. \tag{7.165}$$

When $F = rv_g/f$ and $G = -u_g/fR$, (7.165) expresses the set of combined potential radius and geostrophic azimuth coordinates. The way in which the generalized Clebsch representations (7.36) and (7.37) reduce to (7.165) has been pointed out in section 7.1, although the physics behind such a reduction is still a mystery.

Since we know what forms of the canonical equations we would like to transform to, the easiest way to transform the variational principle is to proceed inversely, i.e.,

$$
\begin{aligned}
&\iiint \left[-\tfrac{1}{2}f\Phi R\delta R + \tfrac{1}{4}fR^2\delta\Phi\right]\sigma_0 d\Omega_0 \\
= &\iiint \left[-\tfrac{1}{2}f(\phi+G)\delta(\tfrac{1}{2}r^2+F) + \tfrac{1}{2}f(\tfrac{1}{2}r^2+F)\delta(\phi+G)\right]\sigma_0 d\Omega_0 \\
= &\iiint \left[-\tfrac{1}{2}f(\phi+2G)\delta(\tfrac{1}{2}r^2) + \tfrac{1}{2}f(\tfrac{1}{2}r^2+2F)\delta\phi - fG\delta F\right]\sigma_0 d\Omega_0 \\
= &\iiint \left[\left(\frac{u_g}{R} - \tfrac{1}{2}f\phi\right) r\delta r + (v_g + \tfrac{1}{4}fr)r\delta\phi + \frac{u_g}{fR}\delta(rv_g)\right]\sigma_0 d\Omega_0 \\
= &\iiint \left[\gamma u_g \delta r + (v_g + \tfrac{1}{2}fr)r\delta\phi\right]\sigma_0 d\Omega_0 + O(R_o^2 f L^2),
\end{aligned}
$$

where $O(R_o^2 fL^2)$ stands for the quadratic term in F and G, the remaining terms form the L_1 dynamics in cylindrical coordinates after comparing with (7.159). We can therefore write our truncated dynamical system in transformed space as

$$
\delta\int d\tau \iiint \left[-\tfrac{1}{2}f\Phi R\frac{\partial R}{\partial\tau} + \tfrac{1}{4}fR^2\frac{\partial\Phi}{\partial\tau} - M^*\right]\sigma_0 d\Omega_0 = 0, \tag{7.166}
$$

where M^* is defined in (7.160).

We now take variations of (7.156) in transformed space, which yields

$$
\delta R: \quad fR\frac{D\Phi}{Dt} = \frac{\partial M^*}{\partial R}, \tag{7.167}
$$

$$
\delta\Phi: \quad f\frac{D\Phi}{Dt} = \frac{\partial M^*}{R\partial\Phi}. \tag{7.168}
$$

Equations (7.167) and (7.168) are the canonically transformed versions of (7.161) and (7.162). They are formally in geostrophic balance when the transformed velocity field is defined as $(U,V) = (DR/Dt, RD\Phi/Dt)$. In comparison with the primitive equation case in the previous section, the essential difference between these canonical equations and (7.120) and (7.123) is that (7.167) and (7.168) form the diagnostic relations between the pressure field and the transformed wind field, while these diagnostic relations can not be attained in (7.120) and (7.123) unless the transient information of Clebsch potentials are supplied.

The balanced model can be constructed in exactly the same way as in Chapter 2. Here we summarize it briefly as follows. From (7.167) and (7.168), we can derive a vorticity equation [refer to (2.66)]. By combining the vorticity equation and the mass continuity equation (7.164), we obtain the potential vorticity equation (2.73) or the potential pseudodensity equation (2.76) which is used as the fundamental predictive equation to get the potential pseudodensity at the next time level. The invertibility principle can be formed through (2.74), from which the predicted potential vorticity or potential pseudodensity is inverted to obtain the balanced wind and mass fields, which then make further prediction possible.

7.3.3 The Hamilton principle and Clebsch transformation associated with the mixed-balance equations on a sphere

In the final effort here, we consider highly curved flow on the sphere. This curvature of the flow is due to the spherical geometry of the Earth. Thus, we shall partition the zonal flow into a gradient balanced flow and the deviation from this balance, and the meridional flow into geostrophic and ageostrophic ones, i.e.,

$$u = u_g + \epsilon u_a, \quad v = \gamma v_g + \epsilon v_a, \tag{7.169}$$

where the zonal gradient momentum u_g, the meridional geostrophic momentum v_g and the curvature parameter γ are all given in (3.36)–(3.38) of Chapter 3.

The Lagrangian for this flow can be written as

$$L = \iiint \left[a\cos\phi(u_g + \Omega a \cos\phi)\frac{\partial \lambda}{\partial \tau} + \gamma v_g a \frac{\partial \phi}{\partial \tau} - M^* \right] \sigma_0 a^2 \cos\phi_0 d\lambda_0 d\phi_0 ds_0 + O(\epsilon) \tag{7.170}$$

where

$$M^* = M + \tfrac{1}{2}(u_g^2 + v_g^2), \tag{7.171}$$

which represents the total energy in the s-coordinate.

One should compare the approximated Lagrangian (7.170) with the full Lagrangian (7.128) which has been used to derive the set of primitive equations. We now prove that this approximated Lagrangian forms a correct variational principle for the mixed balance equations on the sphere that we developed in Chapter 3. By taking variations of (7.170), we have

$\delta\lambda$:

$$\int d\tau \iiint \left[a\cos\phi(u_g + \Omega a\cos\phi)\frac{\partial(\delta\lambda)}{\partial\tau} + a\cos\phi\frac{\partial\lambda}{\partial\tau}\delta u_g + a\frac{\partial\phi}{\partial\tau}\delta(\gamma v_g) \right.$$
$$\left. -\frac{\partial M}{\partial\lambda}\delta\lambda - u_g\delta u_g - v_g\delta v_g \right] \sigma_0 d\Omega_0 + O(\epsilon)$$
$$= \int d\tau \iiint \left[-\frac{\partial}{\partial\tau}(a\cos\phi(u_g + \Omega a\cos\phi)) - \frac{\partial M}{\partial\lambda} \right] \delta\lambda\sigma_0 d\Omega_0 + O(\epsilon) = 0,$$

where it should be noted that the difference between the second and fifth terms, the third and sixth terms result in $O(\epsilon)$ terms, and they can all be put in the small residual.

Because of the arbitrariness of $\delta\lambda$ in this variation, we obtain

$$\frac{\partial u_g}{\partial\tau} - \left(2\Omega\sin\phi + \frac{u_g\tan\phi}{a}\right)v + \frac{\partial M}{a\cos\phi\partial\lambda} = 0. \tag{7.172}$$

Similarly, for the variation $\delta\phi$ of (7.170)

$\delta\phi$:

$$\int d\tau \iiint \left[a\cos\phi\frac{\partial\lambda}{\partial\tau}\delta u_g - a\sin\phi(u_g + 2\Omega a\cos\phi)\frac{\partial\lambda}{\partial\tau}\delta\phi + a\frac{\partial\phi}{\partial\tau}\delta(\gamma v_g) + \gamma v_g a\frac{\partial(\delta\phi)}{\partial\tau} \right.$$
$$\left. -\frac{\partial M}{\partial\phi}\delta\phi - u_g\delta u_g - v_g\delta v_g \right] \sigma_0 d\Omega_0 + O(\epsilon)$$
$$= \int d\tau \iiint \left[-a\sin\phi(u_g + 2\Omega a\cos\phi)\frac{\partial\lambda}{\partial\tau} - \gamma a\frac{\partial v_g}{\partial\tau} - \frac{\partial M}{\partial\phi} \right] \delta\phi\sigma_0 d\Omega_0 + O(\epsilon) = 0,$$

where we have assumed that the change of the curvature distortion factor following particle motion is negligible. By considering the arbitratriness of $\delta\phi$, we have

$$\frac{\partial v_g}{\partial\tau} + \left(2\Omega\sin\phi + \frac{u_g\tan\phi}{a}\right)\frac{u}{\gamma} + \frac{\partial M}{\gamma a\partial\phi} = 0. \tag{7.173}$$

Since the approximation made in the Lagrangian does not change the mass continuity equation (7.125), and the internal and potential energy are left unaltered, one can easily show that the variation of (7.170) in z and the time derivative of (7.125) result in the hydrostatic equation

$$\frac{\partial M}{\partial s} = T, \tag{7.174}$$

and the mass continuity equation

$$\frac{\partial \sigma}{\partial \tau} + \sigma\left(\frac{\partial u}{a\cos\phi\partial\lambda} + \frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial \dot{s}}{\partial s}\right) = 0, \qquad (7.175)$$

respectively. Collecting (7.172)–(7.175), one can see that this set of equations is exactly the set we have derived in Chapter 3 with the combined geostrophic and gradient momentum approximation.

To transform this dynamical system, let us denote the set of reduced Clebsch velocity potentials associated with the vorticity coordinates as

$$\cos^2\Phi = \cos^2\phi + F, \quad \Lambda = \lambda + G. \qquad (7.176)$$

When $F = u_g\cos\phi/\Omega a$ and $G = v_g/2\Omega a\sin\Phi\cos\Phi$, (7.176) expresses the set of combined potential latitude and geostrophic longitude coordinates. The way in which the generalized Clebsch representations (7.61) and (7.62) reduce to (7.176) has been pointed out in section 7.1, although again the physics behind such a reduction is still a mystery.

Since we know what forms of the canonical equations we would like to transform to, the easiest way to transform the variational principle is to proceed inversely, i.e.,

$$\begin{aligned}
& \iiint\left[-\tfrac{1}{2}\Omega a^2\Lambda\delta(\cos^2\Phi) + \tfrac{1}{2}\Omega a^2\cos^2\Phi\delta\Lambda\right]\sigma_0 d\Omega_0 \\
= & \iiint\left[-\tfrac{1}{2}\Omega a^2(\lambda+G)\delta(\cos^2\phi+F) + \tfrac{1}{2}\Omega a^2(\cos^2\phi+F)\delta(\lambda+G)\right]\sigma_0 d\Omega_0 \\
= & \iiint\left[-\tfrac{1}{2}\Omega a^2(\lambda+2G)\delta(\cos^2\phi) + \tfrac{1}{2}\Omega a^2(\cos^2\phi+2F)\delta\lambda - \Omega a^2 G\delta F\right]\sigma_0 d\Omega_0 \\
= & \iiint\left[-\left(\frac{av_g}{2\sin\Phi\cos\Phi} - \tfrac{1}{2}\Omega a^2\lambda\right)\delta(\cos^2\phi) + (a\cos\phi u_g + \tfrac{1}{2}\Omega a^2\cos^{\phi})\delta\lambda\right. \\
& \left. -\frac{v_g}{2\Omega\sin\Phi\cos\Phi}\delta(u_g\cos\phi)\right]\sigma_0 d\Omega_0 \\
= & \iiint\left[a\cos\phi(u_g + \Omega a\cos\phi)\delta\lambda + \gamma v_g a\delta\phi\right]\sigma_0 d\Omega_0 + O(R_o^2 f L^2)
\end{aligned}$$

where $O(R_o^2 f L^2)$ stands for quadratic term in F and G. The remaining terms exactly match the L_1 dynamics presented in (7.170). Therefore, the truncated dynamical system in transformed space can be expressed as

$$\delta\int d\tau\iiint\left[-\tfrac{1}{2}\Omega a^2\Lambda\frac{\partial\cos^2\Phi}{\partial\tau} + \tfrac{1}{2}\Omega a^2\cos^2\Phi\frac{\partial\Lambda}{\partial\tau} - M^*\right]\sigma_0 d\Omega_0 = 0, \qquad (7.177)$$

where M^* is defined in (7.171).

It is straightforward to obtain, after taking the variations of (7.177),

$$\delta\Lambda : \qquad 2\Omega \sin\Phi a \frac{D\Phi}{Dt} = \frac{\partial M^*}{a\cos\Phi\partial\Lambda}, \tag{7.178}$$

$$\delta R : \qquad -2\Omega \sin\Phi a \cos\Phi \frac{D\Lambda}{Dt} = \frac{\partial M^*}{a\partial\Phi}. \tag{7.179}$$

Equations (7.178) and (7.179) are the canonical forms of momentum equations (7.172) and (7.173). They form the important diagnostic relations for balanced dynamics. The general structure of such a balanced dynamical model is discussed in detail in Chapter 3 and outlined in section 7.1 of this chapter.

Chapter 8

CONCLUDING REMARKS

The applicability of balanced dynamics, conceptually, should not be limited to the quasi-straight line type of flows where the two-force balanced system is adequate to capture the essence of these fluid motions, and technically there should be no substantive obstacle to generalizing the quasi-geostrophic and semigeostrophic theories with gradient wind balance while preserving the simple mathematical formulation. Such a balanced theory, however, does not exist to our knowledge. The present study is an attempt to develop a balanced theory that can deal with flows with large curvature. Several aspects centered on this topic have been discussed in this study. In the final chapter here, we will first give a general summary of the current study, then offer our view of future research on this subject.

8.1 Summary of the present study

In Chapters 2 and 3, we have derived a set of filtered equations on the f-plane and on the sphere. The formalism to obtain such balanced systems is very similar to that of semigeostrophic theory. Through a Rossby number analysis, we impose a combined geostrophic-gradient momentum approximation in the primitive equations. The canonical transformation of this set of approximate equations leads to the universal formulation of balanced dynamics, i.e., one predictive equation for the potential pseudodensity (the reciprocal of potential vorticity) and one invertibility principle to diagnose the balanced wind and mass fields. These equations (either in physical space or in transformed space) may be referred to as the mixed-balance equations, and the theoretical framework may correspondingly be referred to as the mixed-balance theory.

The terminology "universal formulation" implies that all balanced dynamic models, no matter how complicated, should eventually take a similar formulation. This is understandable because the implementation of the balance assumption in the dynamic system effectively reduces the number of prognostic equations to one (so that there is only one class of wave motions), and the remaining diagnostic equations can be reformulated and combined to yield one inversion operator. The choice of the advected substance (the predictive quantity) can, technically speaking, be arbitrary. However, if we choose this quantity in such a way that it carries both dynamic and thermodynamic information, and is conservative, it will give the simplest closed form of the dynamics. This quantity is virtually embedded in the principle derived by Rossby (1940) and Ertel (1941), i.e., the potential vorticity. Thus, PV becomes the substance to be advected and carried around by the fluid motion. Under certain balanced assumptions, a snapshot of the distribution of this substance provides all the essential meteorological information. For balanced models whose universal formulation is achieved by coordinate transformation, it turns out that a more amenable quantity to be advected is the potential pseudodensity (Schubert *et al.* 1989).

Table 8.1 summarizes the different balanced models in terms of their formulations: the fundamental predictive equation and the invertibility principle; their predictive quantity and their balance type. Needless to say, all these models have the same formulation (the universal formulation we discussed previously). Thus, the solution procedures for these models are all the same, i.e., the solution of one time evolution equation (mixed initial-boundary value problem) and one elliptic equation (pure boundary value problem). However, as the balanced models become more general, the degree of nonlinearity of the elliptic equation (invertibility principle) becomes higher. In comparison, the mixed-balance model has a nearly identical structure to that of the semigeostrophic model. Their invertibility principles have cubic nonlinearity.

The mixed-balance theory explored in this study has nice physical properties in the sense that it has conservation principles for total energy, angular momentum and potential vorticity. In addition, it preserves the fully three-dimensional vorticity equation. Table

1. Nondivergent barotropic model

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \beta v = 0,$$

$$\zeta = \nabla^2 \psi$$

Predictive quantity: vorticity. Balance type: two-dimensional incompressible.

2. Quasi-geostropic model

$$\frac{\partial q}{\partial t} + u_g\frac{\partial q}{\partial x} + v_g\frac{\partial q}{\partial y} = 0,$$

$$q = \nabla^2\Phi + \frac{\partial}{\rho \partial z}\left(\frac{\rho f^2}{N^2}\frac{\partial \Phi}{\partial z}\right).$$

Predictive quantity: quasi-geostrophic potential vorticity. Balance type: geostrophic.

3. Balanced equation model

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \beta v + f\nabla^2\chi = 0,$$

$$\nabla^2[\Phi + \tfrac{1}{2}(\nabla\psi)^2] = \nabla \cdot [(f+\zeta)\nabla\psi].$$

Predictive quantity: vorticity. Balance type: nonlinear.

4. Semigeostrophic model

$$\frac{\partial \sigma^*}{\partial T} + \frac{\partial(u_g\sigma^*)}{\partial X} + \frac{\partial(v_g\sigma^*)}{\partial Y} + \frac{\partial(\dot{\theta}\sigma^*)}{\partial \Theta} = 0,$$

$$\frac{\partial(x,y,\Pi)}{\partial(X,Y,\Theta)} + \Gamma\sigma^* = 0.$$

Predictive quantity: potential pseudodensity. Balance type: geostrophic.

5. Mixed-balanced model

$$\frac{\partial \sigma^*}{\partial T} + \frac{\partial(RU\sigma^*)}{R\partial R} + \frac{\partial(V\sigma^*)}{R\partial \Phi} + \frac{\partial(\dot{S}\sigma^*)}{\partial S} = 0,$$

$$\frac{\partial(\frac{1}{2}r^2,\phi,p)}{\partial(\frac{1}{2}R^2,\Phi,S)} + \sigma^* = 0.$$

Predictive quantity: potential pseudodensity. Balance type: combined geostrophic-gradient.

Table 8.1: Formulation, predictive quantity and balance type for balanced models.

Model	Energy	PV	Vorticity	Momentum
Nondivergent barotropic	yes	yes[1]	3rd component	yes
Quasi-geostrophic	yes	yes	3rd component[2]	yes
SW balance equation	no	yes	3rd component	no
Isob. balance equation	yes	no	3rd component	no
Isen. balance equation	no	yes	full	no
Semigeostrophic	yes	yes	full	yes
Mixed-balance	yes	yes	full	yes

Table 8.2: Physical properties associated with different balanced models.

8.2 compares these physical properties for each balanced model. In this table, the "SW balance equation" denotes the balance equation derived from the approximation of the shallow water primitive equations; the " Isob. balance equation" denotes the balance equation derived from the approximation of the fully stratified primitive equations in pressure coordinate (the Charney-Bolin type of balance equation); the " Isen. balance equation" denotes the balance equation derived from the fully stratified primitive equations in isentropic coordinate. Footnote 1 in the first line of this table indicates that there is no distinction between vorticity and potential vorticity in a nondivergent barotropic model. Footnote 2 in the second line indicates that although one may derive a three-dimensional vorticity equation for QG system, only the third component of the vorticity equation is used to close the dynamic system.

The mixed-balance theory also possesses consistent asymptotic properties. First, as the local radius of curvature becomes infinitely large, the mixed-balance equations reduce to the semigeostrophic equations. Secondly, in the axisymmetric limit (or zonally symmetric limit), the mixed-balance model becomes the Eliassen balanced vortex model (or zonally symmetric model). In Chapter 7, we also demonstrated that the Hamiltonian structure associated with the mixed-balance equations is also constructable. Its canonical transformation is closely related to the reduced form of Clebsch velocity representation.

Although the full application of the mixed-balance theory to a particular atmospheric phenomenon has not yet been attempted here, we do compute the eigensolutions of the mixed-balance equations both on the f-plane and on the sphere (Chapter 5). For f-plane theory, we idealize a tropical cyclone by a Rankine vortex, and use this vortex as the basic state. The computed eigenfrequencies and eigenfunctions are compared with those from the primitive equation model and from the nondivergent barotropic model. The results suggest that the eigenmodes depicted in the mixed-balance equations resemble the Rossby modes in the linear manifold of the primitive equation model, and that the linear solutions from the mixed-balance model retain reasonable accuracy in comparison with those from the PE model. The analytical solution obtained from the nondivergent barotropic model differs sharply from those obtained from the mixed-balance model and therefore from the PE model. We conjectured that this difference may be largely due to the difference between the divergent model and the nondivergent model. Different vortex intensities are adopted in the calculations with the observational values for the tangential wind and the horizontal scale of the vortex. Three vortices with different intensity are used to represent three development stages of a tropical cyclone, i.e., tropical depression, tropical storm and hurricane. We find that the Rossby wave frequency is strongly dependent upon the vortex rotational rate. As the vortex spins up, the Rossby wave frequencies are substantially enhanced. By examining the full spectrum of eigenfrequencies from the primitive equation calculation, we find a class of Rossby waves with such high frequencies that their dispersion curves overlap those of inertia-gravity waves. The eigensolutions of the spherical version of the mixed-balance equations are also computed, and they compare quite well with the primitive equation results of Longuet-Higgins (1968).

A converging view of shear instability from the mixed-balance theory and from the previous balanced theories is discussed in Chapter 6. The linearized mixed-balance equations naturally lead to a stability theorem of the Charney-Stern type. For a circular flow on an f-plane, the necessary condition for combined barotropic and baroclinic instability is that the radial gradient of potential pseudodensity changes sign somewhere in the domain. For a circular flow on the spherical earth, the necessary condition for instability to occur is that the meridional gradient of potential pseudodensity changes sign somewhere in the domain. The difference between these newly derived stability theorems and previous ones is that the curvature of the basic flow has been taken into account. This curvature effect, apparently, does not alter the general stability statement. As a byproduct of the above analysis, we also derived the generalized Eliassen-Palm theorems associated with the mixed-balance equations on the f-plane and on the sphere.

Since the Kelvin wave is invisible on a potential vorticity map, the new balanced model developed here may not be suitable for study of some tropical weather phenomena such as the Madden-Julian oscillation or the quasi-biennial oscillation. On the other hand, if we consider a flow with a predominent zonal component, the mixed-balance equations on a sphere (see Chapter 3) reduce to Gill's long-wave approximate system (Gill, 1980; Stevens *et al.*, 1990) by a scaling argument. Under this approximation, Kelvin waves are included. This issue is worthy of further investigation.

8.2 Directions for future research

Since circular flows and other highly curved flows are common patterns of fluid motion on the rotating spherical planet (see Figure 1.1 of Chpter 1), the theory developed in this study can find many potential applications. For example, we may use the mixed-balance theory to study tropical cyclones, midlatitude synoptic disturbances, polar lows and even some mesoscale convective systems. For these weather systems, traditional balanced models such as the quasi-geostrophic or semigeostrophic equations may not describe the fluid motions correctly, or at least accurately (Snyder *et al.* 1991). The primitive equation model, on the other hand, may not give clear physical insights due to its generality. From Table 8.1 we can see that the mixed-balance model has an almost identical formulation to that of the semigeostrophic model. Therefore, the solution technique for the semigeostrophic model can be utilized to solve the mixed-balance equations with only minor modification.

Another class of problems for application of mixed-balance theory is that solved by the symmetric models. These problems include the Hadley circulation [which has been solved by using a zonally symmetric model, e.g., Hack *et al.* (1989), Schubert *et al.* (1991)] and hurricane circulation [which has been solved by using the axisymmetric balanced model, e.g, Ooyama (1969), Schubert and Alworth (1987)]. Although these atmospheric phenomena are dominated by low-wavenumber motions, the eddy motions still contribute importantly to the total dynamical picture. [e.g., the eddy transports of momentum and energy in the Hadley circulation (Lorenz, 1967) and the spiral bands in a hurricane (Guinn and Schubert, 1993)]. A reformulation of the mixed-balance model other than the potential vorticity (potential pseudodensity) formulation is possible. The new formulation most likely involves a pair of transversal circulation equations with a predictive equation either for one of the horizontal momentums or for entropy. This methodology has been discussed in Hoskins and Draghici (1977) in the context of semigeostrophic equations. Such a formulation for the balanced equations presented in Chapter 3 [(3.31)–(3.34)] may have a potential application to the physical problem of the coupled Hadley and Walker circulations.

The class of high frequency Rossby waves identified in Chapters 4 and 5 may be of importance both theoretically and practically. First, the existence of these high frequency Rossby waves may alter the concept that Rossby waves are always low-frequency motions. Secondly, these high frequency Rossby waves revealed in our mixed-balance model raise the question of what the balanced (or filtered) models mean. Do the balanced models filter all the fast modes and retain the slow ones? When some of the Rossby frequencies are as high as those of inertia-gravity waves, what do the balanced models filter? Is there such a thing as selective filtering (filtering the wave types rather than frequency ranges)? We speculate that the occurrence of these high frequency oscillations in the balanced model may be closely related to the fuzziness of the slow manifold discussed by Warn and Menard (1986), Lorenz and Krishnamurthy (1987) and McIntyre and Norton (1992). In these studies, they argue that a true slow manifold that is completely devoid of high frequency oscillations may not exist. However, since the fast waves presented in our balanced model are strongly dependent upon the rotational rate of a spinning vortex, they possess Rossby-like features. In this sense, they may be different from the slaved gravity-like modes associated with nonlinear normal mode initialization, and from the spontaneously emitted gravity waves at nonzero Froude number and Rossby number. Furthermore, as the dispersion curves of the high frequency Rossby waves intersect those of inertia-gravity waves, there is no longer a clear separation of fast manifold and slow manifold [like those presented in Matsuno (1966) and in Longuet-Higgins (1968)]. This situation may have profound implications on the stability analysis. For example, in Ripa's theorem, if we interpret one of his stability conditions as limiting the phase lock between the Rossby waves, and the other one as limiting the phase lock between the inertia-gravity waves, we then need a more general condition to limit the phase lock between the Rossby waves and the inertia-gravity waves in the situations considered above. This mechanism has been conjectured by Sakai (1989) as a new type of ageostrophic instability caused by a resonance between Rossby waves and gravity waves. In order to understand these questions, we need to extend the current study by conducting more normal mode analyses with different basic wind profiles.

In association with the theoretical results obtained in Chapter 6, we expect that some more applied normal mode instability problems can be investigated in the future. We have developed the barotropic versions of the eigensolvers for the mixed-balance model both on the f-plane and on the sphere. With these solvers we can study instability problems involving curved flows such as hurricanes, extratropical cyclones and polar lows, by employing observed basic wind profiles. The results from these studies can be compared with similar studies using the nondivergent barotropic model, the quasi-geostrophic model and the primitive equation model (Staley and Gall, 1979; Gent and McWilliams, 1986; Flatau and Stevens, 1989). We can further extend these eigensolvers to include vertical discretization so that we can study the nonseparable baroclinic instability, or the combined barotropic and baroclinic instabililty problems (McIntyre, 1970; Kuo, 1978; Moore and Peltier, 1989, 1990). The theorems derived in Chapter 6 can be used as guides to aid understanding of this class of instability problems.

There are still possibilities to further generalize the mixed-balance theory with even higher order balanced approximations. For example, in parallel with the formulations of

QG, SG and the mixed-balance models, can we use the nonlinear balanced wind to replace the advected total wind to obtain a more general balanced dynamic system? This general balanced model could possibly treat any kind of balanced flow on the spherical earth. It preserves the simple formulation of all balanced models (see discussion in the previous section), with a more accurate but more complicated PV inversion operator. The current mixed-balance model may be a special case of this general balance theory. In Chapter 7, we have shed some light on such a possible generalization. It seems that the discovery of this theory may crucially depend on the full understanding of Clebsch transformations. A probable way to proceed may be to find a properly approximated Hamilton's principle, and to combine this principle with the properly simplified Clebsch representation of velocity.

We may also consider theoretical extensions to the stability theories derived in Chapter 6. Future generalizations may lie in two aspects. First, the linear stability theorems can be generalized to the finite-amplitude results (see Table 6.1 of Chapter 6). This generalization may have some theoretical significance, as shown by McIntyre and Shepherd (1987) and Shepherd (1988, 1989) for the two-dimensional incompressible flow and quasi-geostrophic flow. Secondly, by using the Casimir approach, it may be possible to rederive a generalized wave-activity relation for the mixed-balance system so that the wave-activity density can be expressed in terms of Eulerian quantities. A similar generalization for the primitive equation system has been found by Haynes (1988). This generalization may have some practical significance. Combining this generalized wave-activity relation and the mean state momentum equations, one can quantitatively study wave-mean interaction problems.

REFERENCES

Abarbanel, H. D. I., D. D. Holm, J. E. Marsden and T. S. Ratiu, 1986: Non-linear stability analysis of stratified fluid equilibria. *Phil. Trans. Roy. Soc. London*, **A318**, 349-409.

Allen, J. S., J. A. Barth and P. A. Newberger, 1990a: On intermediate models for barotropic continental shelf and slope flow fields. Part I: Formulation and comparison of exact solutions. *J. Phys. Oceanogr.*, **20**, 1017–1042.

Allen, J. S., J. A. Barth and P. A. Newberger, 1990b: On intermediate models for barotropic continental shelf and slope flow fields. Part III: Comparison of numerical model solutions in periodic channels. *J. Phys. Oceanogr.*, **20**, 1949–1973.

Arnold, V. I., 1965: Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid. *Dokl. Akad. Nauk. SSSR* **162**, 975-978. (English transl.: *Soviet Math.* **6**, 773-777 (1965).)

Arnold, V. I., 1966a: On an apriori estamate in the theory of hydrodynamical stability. *Izv. Vyssh. Uchebn. Zaved. Matematika* **54**, no. 5, 3-5. (English transl.: *American Math. Soc. Transl. Series 2*, **79**, 267-269 (1969).)

Andrews, D. G., and M. E. McIntyre, 1976: A finite-amplitude Eliassen-Palm theorem in isentropic coordinates. *J. Atmos. Sci.*, **40**, 1877–1883.

Andrews, D. G., and M. E. McIntyre, 1978: Generalized Eliassen-Palm and Charney-Drazin theorem for waves on axisymmetric flow in compressible atmosphere. *J. Atmos. Sci.*, **35**, 175–185.

Andrews, D. G., 1983: A finite-amplitude Eliassen-Palm theorem in isentropic coordinates. *J. Atmos. Sci.*, **40**, 1877–1883.

Bakelman, I. Ya., 1957: Generalized solutions of Monge-Ampere equations. Doklady Akad. Nauk SSSR, Vol. 114, No. 6, pp. 1143–1145 (Russian).

Bakelman, I. Ya., 1958: On the theory of Monge-Ampere equations. Vestnik LGU, No. 1 (Russian).

Bakelman, I. Ya., 1958: Regularity of solutions of Monge-Ampere equations. Leningrad, Uchenye zapiski kafedry matematiki Pedinstituta, 166 (Russian).

Barth, J. A., J. S. Allen and P. A. Newberger, 1990: On intermediate models for barotropic continental shelf and slope flow fields. Part II: Comparison of numerical model solutions in doubly periodic domains. *J. Phys. Oceanogr.*, **20**, 1044–1076.

Benjamin, T. B., 1984: Impulse, flow force and variational principles. *IMA J. Appl. Math.*, **32**, pp. 3–68.

Bolin, B., 1955: Numerical forecasting with the barotropic model. *Tellus*, **7**, 27-49.

Bolin, B., 1956: An improved barotropic model and some aspects of using the balance equations for three-dimensional flow. *Tellus*, **8**, 61-75.

Bretherton, F. P., 1966: Critical layer instability in baroclinic flows. *Quart. J. Roy. Meteor. Soc.*, **92**, 325–334.

Buzzi, A., A. Trevisan and G. Salustri, 1981: Internal frontogenesis: A two-dimensional model in isentropic, semi-geostrophic coordinates. *Mon. Wea. Rev.*, **109**, 1053–1060.

Charney, J. G., 1948: On the scale of atmospheric motions. *Geofys. Publ.*, **17**, No. 2, 1–17.

Charney, J. G., 1955: The use of primitive equations of motion in numerical prediction. *Tellus*, **7**, 27-49.

Charney, J. G., 1962: Integration of the primitive and balance equations. *Proc. Int. Symp. Numerical Weather Prediction*, Meteor. Soc. Japan, Tokyo, 131-152.

Charney, J. G., and M. E. Stern, 1962: On the stability of internal baroclinic jets in a rotating atmosphere. *J. Atmos. Sci.*, **19**, 159–172.

Courant, R., and D. Hilbert, 1966: *Method of Mathematical Physics*, volume II. Wiley-Interscience, 830pp.

Craig, G. C., 1991: A three dimensional generalization of Eliassen's balanced vortex equations derived from Hamilton's principle. *Quart. J. Roy. Meteor. Soc.*, **117**, 435–448.

Davies, H. C., C. Schär and W. Wernli, 1991: The palette of fronts and cyclones within a baroclinic wave development. *J. Atmos. Sci.*, **48**, 1666–1689.

Dikii, L. A., 1966: On the asymptotic solution of Laplace's tidal equation. *Dokl. Acad. Nauk SSSR, Atoms. Ocean. Phys.*, **1**, 469–489; *Izv. Acad. Sci. USSR, Atmos. Ocean. Phys.*, pp. 275–286.

Eliassen, A., 1948: The quasi-static equations of motion. *Geofys. Publ.*, **17**(3).

Eliassen, A., 1952: Slow thermally or frictionally controlled meridional circulation in a circular vortex. *Astrophys. Norv.*, **5**, 60pp.

Eliassen, A., 1983: The Charney-Stern theorem on barotropic and baroclinic instability. *Pure Appl. Geophys.*, **121**, 563–572.

Elsberry, R. L., W. M. Frank, G. J. Holland, J. D. Jarrell and R. L. Southern, 1987: *A Global View of Tropical Cyclones.* Int. Workshop on Tropical Cyclones, Bangkok, Thailand.

Ertel, H., 1942: A new hydrodynamical vorticity equation. English translation by W. H. Schubert from Ein neuer hydrodynamischer Wirbelsätz. *Meteorologische Zeitschrift*, **59**, 22-30.

Fjørtoft, R., 1950: Application of integral theorems in deriving crieria of stability for laminar flows and for the baroclinic circular vortex. *Geofys. Publ.*, **17**, No. 6, 52 pp.

Flatau, M., and D. E. Stevens, 1989: Barotropic and inertial instabilities in the hurricane outflow layer. *Geophys. Astrophys. Fluid Dyn.*, **47**, 1–18.

Fulton, S. R., and W. H. Schubert, 1991: Surface frontogenesis in isentropic coordinates. *J. Atmos. Sci.*, **48**, 2534–2541.

Fulton, S. R., and R. K. Taft, 1991: Multigrid solvers for the three-dimensional semi-geostrophic equations. *Preprints from the 9th Conference on NWP*, Amer. Meteor. Soc., 309–315.

Gent, P. R., and J. C. McWilliams, 1986: The instability of barotropic circular vortices. *Geophys. Astrophys. Fluid Dyn.*, **35**, 209–233.

Gill, A. E., 1980: Some simple solutions for heat-induced tropical circulation. *Quart. J. Roy. Meteor. Soc.*, **106**, 447–462.

Gill, A. E., 1981: Homogeneous intrusions in a rotating stratified fluid. *J. Fluid Mech.*, **103**, 275–295.

Guinn, T. A., and W. H. Schubert, 1993: Hurricane spiral bands. *Submitted to J. Atmos. Sci.*

Hack, J. J., W. H. Schubert, D. E. Stevens, and H.-C. Kuo, 1989: Response of the Hadley circulation to convective forcing in the ITCZ. *J. Atmos. Sci.*, **46**, 2957–2973.

Haurwitz, B., 1940: The motion of atmospheric disturbances on the spherical earth. *J. Marine Res.*, **3**, 254–267.

Hawkins, H. F., and S. M. Imbembo, 1976: The structure of a small, intense hurricane, Inez 1966. *Mon. Wea. Rev.*, **104**, 418–442.

Haynes, P. H., and M. E. McIntyre, 1987: On the evolution of vorticity and potential vorticity in the presence od diabatic heating and frictional or other forces. *J. Atmos. Sci.*, **44**, 828–841.

Haynes, P. H., 1988: Forced, dissipative, generalizations of finite amplitude wave-activity conservation relations for zonal and nonzonal basic flows. *J. Atmos. Sci.*, **45**, 2352–2362.

Haynes, P. H., and M. E. McIntyre, 1990: On the conservation and impermeability theorems for potential vorticity. *J. Atmos. Sci.*, **47**, 2021–2031.

Heckley, W. A., and B. J. Hoskins, 1982: Baroclinic waves and frontogenesis in a nonuniform potential vorticity semi-geostrophic model. *J. Atmos. Sci.*, **39**, 1999–2016.

Hertenstein, R. F. A., and W. H. Schubert, 1991: Potential vorticity anomalies associated with squall lines. *Mon. Wea. Rev.*, **119**, 1663–1672.

Hoskins, B. J., and F. P. Bretherton, 1972: Atmospheric frontogensis models: Mathematical formulation and solution. J. Atmos. Sci., **29**, 11-37.

Hoskins, B. J., 1975: The geostrophic momentum approximation and the semi-geostrophic equations. *J. Atmos. Sci.*, **32**, 233–242.

Hoskins, B. J., and I. Draghici, 1977: The forcing of ageostrophic motion according to the semi-geostrophic equations and in an isentropic coordinate model. *J. Atmos. Sci.*, **34**, 1859–1867.

Hoskins, B. J., and N. V. West, 1979: Baroclinic waves and frontogenesis. Part II: Uniform potential vorticity jet flows–cold and warm fronts. *J. Atmos. Sci.*, **36**, 1663–1680.

Hoskins, B. J., M. E. McIntyre and A. W. Robertson, 1985: On the use and significance of isentropic potential vorticity maps. *Quart. J. Roy. Meteor. Soc.*, **111**, 877–946.

Hough, S. S., 1898: On the application of harmonic analysis to the dynamical theory of the tides. II: On the general integration of Laplace's tidal equations. *Phil. Trans. Roy. Soc. London*, **A191**, 139-185.

Kasahara, A., 1974: Various vertical coordinate systems used for Numerical weather prediction. *Mon. Wea. Rev.*, **102**, 509–522.

Kuo, H. L., 1949: Dynamic instability of two-dimensional nondivergent flow in a barotropic atmosphere. *J. Meteor.*, **6**, 105–122.

Kuo, H. L., 1978: A two-layer model study of the combined barotropic and baroclinic instability in the tropics. *J. Atmos. Sci.*, **35**, 1840–1860.

Lamb, H., 1932: *Hydrodynamics.* Sixth edition. Dover Publications, 738 pp.

Lanczos, C., 1970: *The Variational Principles of Mechanics.* Fourth Edition. Dover Publications, 418 pp.

Leith, C. E., 1980: Nonlinear normal mode initialization and quasi-geostrophic theory. *J. Atmos. Sci.*, **37**, 958–968.

Lin, C. C., 1963: Hydrodynamics of helium II. *Proc. Int. Sch. Phys. XXI.* pp 93-146. New York: academic.

Lindzen, R. D., 1967: Planetary waves on β-planes. *Mon. Wea. Rev.*, Vol. 95, **7**, 441–451.

Longuet-Higgins, M. S., 1968: The eigenfunctions of Laplace's tidal equations over a sphere. *Phil. Trans. Roy. Soc. London*, **A262**, 511-607.

Lorenz, E. N., 1955: Available potential energy and the maintenance of the general circulation. *Tellus*, **7**, 157-167.

Lorenz, E. N., 1960: Energy and numerical prediction. *Tellus*, **12**, 364-373.

Lorenz, E. N., 1960: Maximum simplification of the dynamic equations. *Tellus*, **12**, 243-254.

Lorenz, E. N., 1967: *The Nature and Theory of The General Circulation of the Atmosphere*. WMO Publication, 161 pp.

Lorenz, E. N., 1980: Attractor sets and quasi-geostrophic equilibrium. *J. Atmos. Sci.*, **37**, 1685-1699.

Lorenz, E. N., 1986: On the existence of a slow manifold. *J. Atmos. Sci.*, **43**, 1685-1699.

Lorenz, E. N., and V. Krishnamurthy, 1987: On the nonexistence of a slow manifold. *J. Atmos. Sci.*, **44**, 2940-2950.

Machenhauer, B., 1977: On the dynamics of gravity oscillations in a shallow water model, with application to normal mode initialization. *Beitr. Phys. Atmos.*, **50**, 253–271.

Magnusdottir, G., 1989: Generalized semigeostrophic theory. A Ph.D dissertation, Dept. of Atmos. Sci., Colorado State Univ. No. 459.

Magnusdottir, G., and W. H. Schubert, 1990: The generalization of semi-geostrophic theory to the β-plane. *J. Atmos. Sci.*, **47**, 1714–1720.

Magnusdottir, G., and W. H. Schubert, 1991: Semigeostrophic theory on the hemisphere. *J. Atmos. Sci.*, **48**, 1449–1456.

Margules, M., 1893: Lufftbewegungen in einer rotierenden Sphäroidschale. Theil II, *Sitzungsberichte der Mathematischen Naturwissenschaftlichen Klasse*, Kaiserliche Academie der Wissenschaften, Wien, vol. 102, **1**, pp. 11-56.

Matsuno, T., 1966: Quasi-geostrophic motions in the equatorial area. *J. Meteor. Soc. Japan*, **44**, 25-42.

McIntyre, M. E., 1970: On the non-separable baroclinic parallel flow instability problem. *J. Fluid Mech.*, **40**, 273–306.

McIntyre, M. E., and T. G. Shepherd, 1987: An exact local conservation theorem for finite-amplitude disturbances to non-parallel shear flows, with remarks on Hamiltonian structure and on Arnol'd's stability theorems. *J. Fluid Mech.*, **181**, 527–565.

McIntyre, M. E., and W. A. Norton, 1990: Dissipative wave-mean interactions and the transport of vorticity or potential vorticity. *J. Fluid Mech.*, **212**, 403–435.

McIntyre, M. E., and W. A. Norton, 1990: Potential-vorticity inversion on a hemisphere. *J. Atmos. Sci.*, to be submitted.

McWilliams, J. C., and P. R. Gent, 1980: Intermediate models of planetary circulations in the atmosphere and ocean. *J. Atmos. Sci.*, **37** 1657–1678.

Montgomery, M. T., and B. F. Farrell, 1990: Dry surface frontogenesis arising from interior potential vorticity perturbations in a semigeostrophic model. *J. Atmos. Sci.*, **47**, 2837–2852.

Montgomery, M. T., and B. F. Farrell, 1991: Moist surface frontogenesis associated with interior potential vorticity anomalies in a semigeostrophic model. *J. Atmos. Sci.*, **48**, 343–367.

Moore, G. W. K., and W. R. Peltier, 1989: Non-separable baroclinic instability. Part I: Quasi-geostrophic dynamics. *J. Atmos. Sci.*, **46**, 57–78.

Moore, G. W. K., and W. R. Peltier, 1990: Non-separable baroclinic instability. Part II: Primitive equations dynamics. *J. Atmos. Sci.*, **47**, 1223–1242.

Moura, A. D., 1976: The eigensolutions of the linearized balance equations over a sphere. *J. Atmos. Sci.*, **33**, 877–907.

Ooyama, K., 1969: Numerical simulation of the life cycle of tropical cyclones. *J. Atmos. Sci.*, **26**, 3–40.

Phillips, N. A., 1966: The equation of motion for a shallow rotating atmosphere and the traditional approximation. *Beitr. Phys. Atmos.*, **23**, 626–628.

Pogorelov, A. V., 1964: *Monge-Ampere Equations of Elliptic type.* Published by Noordhoff, Ltd., The Netherlands, 116 pp.

Rayleigh, Lord, 1880: On the stability or instability of certain fluid motions. *Proc. Lond. Math. Soc.*, **11**, 57–70.

Raymond, D. J., and H. Jiang, 1990: A theory for long-lived mesoscale convective systems. *J. Atmos. Sci.*, **47**, 3067–3077.

Ripa, P., 1983: General stability conditions for zonal flow in a one-layer model on the β-plane or the sphere. *J. Fluid Mech.*, **126**, 463–489.

Ripa, P., 1991: General stability conditions for a multi-layer model. *J. Fluid Mech.*, **222**, 119–137.

Rossby, C.-G., 1939: Relation between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semi-permanent centers of action. *J. Marine Res.*, **2**(1), 38–55.

Rossby, C.-G., 1940: Planetary flow patterns in the atmosphere. *Quart. J. Roy. Meteor. Soc.*, **66**, Suppl., 68–87.

Saarloos, W. van, 1981: A canonical transformation relating the Lagrangian and Eulerian description of ideal hydrodynamics. *Physica*, **108A**, 557-566.

Sakai, S., 1989: Rossby-Kelvin instability: a new type of ageostrophic instability caused by a resonance between Rossby waves and gravity waves. *J. Fluid Mech.*, **202**, 149–176.

Salmon, R., 1982: Hamilton's principle and Ertel's theorem. *Am. Inst. Phys. Proc.*, **88**, 127–135.

Salmon, R., 1983: Practical use of Hamilton's principle. *J. Fluid Mech.*, **132**, 431–444.

Salmon, R., 1985: New equations for nearly geostrophic flow. *J. Fluid Mech.*, **153**, 461–477.

Salmon, R., 1988: Hamiltonian fluid mechanics. *Ann. Rev. Fluid Mech.*, **20**, 225–256.

Schär, C., and H. C. Davies, 1990: An instability of mature cold fronts. *J. Atmos. Sci.*, **47**, 929–950.

Schubert, W. H., and J. J. Hack, 1983: Transformed Eliassen balanced vortex model. *J. Atmos. Sci.*, **40**, 1571–1583.

Schubert, W. H., and B. T. Alworth, 1987: Evolution of potential vorticity in tropical cyclones. *Quart. J. Roy. Meteor. Soc.*, **113**, 147–162.

Schubert, W. H., S. R. Fulton and R. F. A. Hertenstein, 1989: Balanced atmospheric response to squall lines. *J. Atmos. Sci.*, **46**, 2478–2483.

Schubert, W. H., D. E. Stevens, P. E. Ciesielski and H.-C. Kuo, 1991: The evolution of potential vorticity in the Hadley circulation. *J. Atmos. Sci.*, **48**, 1493–1509.

Schubert, W. H., and G. Magnusdottir, 1991: Clebsch transformations of the primitive equations. *Preprints from the 8th Conference on Atmos. and Ocean Wave and Stability*, Amer. Meteor. Soc., 141–142.

Seliger, R. L., and G. B. Whitham, 1968: Variational principles in continuum mechanics. *Proc. Roy. Soc. London*, **A305**, 1-25.

Sheets, R. C., 1980: Some aspects of tropical cyclone modification. *Aust. Meteor. Mag.*, **27**, 259–280.

Shepherd, T. G., 1988: Nonlinear saturation of baroclinic instability. Part I: The two layer model. *J. Atmos. Sci.*, **45**, 2014–2025.

Shepherd, T. G., 1988: Nonlinear saturation of baroclinic instability. Part II: Continuously stratified fluid. *J. Atmos. Sci.*, **46**, 888–907.

Shepherd, T. G., 1990: Symmetries, conservation laws, and Hamiltonian structure in geophysical fluid dynamics. *Advances in Geophysics*, **32**, pp. 287–338.

Shutts, G. J., and A. J. Thorpe, 1978: Some aspects of vortices in rotating, stratified fluids. *Pure. Appl. Geophys.*, **116**, 993–1006.

Shutts, G. J., 1980: Angular momentum coordinates and their use in zonal, geostrophic motion in a hemisphere. *J. Atmos. Sci.*, **37**, 1126–1132.

Shutts, G. J., 1989: Planetary semi-geostrophic equations derived from Hamilton's principle. *J. Fluid Mech.*, **208**, 545–573.

Snyder, C., C. Skamarock and R. Rotunno, 1991: A comparison of primitive equation and semigeostrophic simulations of baroclinic waves. *J. Atmos. Sci.*, **48**, 2179–2194.

Staley, D. O., and R. L. Gall, 1979: Barotropic instability in a tornado vortex. *J. Atmos. Sci.*, **36**, 973–981.

Stevens, D. E., and P. E. Ciesielski, 1984: A global model of linearized atmospheric perturbations: model description. Atmospheric Science Paper No. 377, Colorado State University.

Stevens, D. E., H. -C. Kuo, W. H. Schubert and P. E. Ciesielski, 1990: Quasi-balanced dynamics in the tropics. *J. Atmos. Sci.*, **47**, 2262–2273.

Taylor, G. I., 1915: Eddy motion in the atmosphere. *Phil. Trans. Roy. Soc. London*, **A115**, 1-26.

Thomson, W., 1879: On gravitational oscillations of rotating water. *Proc. Roy. Soc. Edin.*, **x.92**, [*papers*, iv. 141].

Thomson, W., 1880: On the vibrations of a columnar vortex. *Phil. Mag.*(5), **x.155**, [*papers*, iv. 152].

Vallis, G. K., 1992: Mechanisms and parameterizations of geostrophic adjustment and a variational approach to balanced flow. *J. Atmos. Sci.*, **49**, 1144–1160.

Wallace, J. M., and P. V. Hobbs, 1977: *Atmospheric Science.* Academic Press, 467 pp.

Warn, T., and R. Menard, 1986: Nonlinear balance and gravity-inertial wave saturation in a simple atmospheric model. *Tellus*, **38A**, 285-294.

Zhou, Y. B., 1979: *Theoretical Mechanics.* People's Edu. Publish, China, 341 pp.

Appendix A

TRANSFORMATION OF THE MOMENTUM EQUATIONS

In this appendix, we will prove that the horizontal momentum equations with the combined geostrophic-gradient momentum approximation can be transformed to their canonical forms. In the first section, we will use a set of combined potential radius and geostrophic azimuth coordinates to transform the mixed-balance equations on an f-plane. The transformation of mixed-balance equations on the sphere by a set of combined potential latitude and geostrophic longitude coordinates is discussed in the second section.

A.1 The mixed-balance momentum equations on an f-plane

The easist way to transform (2.37) and (2.38) is to work backward from (2.64) and (2.65):

$$fR\frac{D\Phi}{Dt} = \frac{\partial M^*}{\partial R}, \tag{A.1}$$

$$-f\frac{DR}{Dt} = \frac{\partial M^*}{R\partial\Phi}. \tag{A.2}$$

By using the first entry of (2.61) and substituting the geostrophic azimuth coordinate (2.56), we can rewrite (A.1) as

$$fR\frac{D}{Dt}\left(\phi - \frac{u_g}{fR}\right) = \frac{\partial M}{\gamma\partial r} + \frac{u_g^2}{R},$$

or,

$$\frac{R}{r}fv - \frac{Du_g}{Dt} + \frac{u_g}{R}\frac{DR}{Dt} = \frac{\partial M}{\gamma\partial r} + \frac{u_g^2}{R}.$$

The second line is obtained by directly calculating the derivative. Substituting (A.2) for DR/Dt, we have

$$\frac{R}{r}fv - \frac{Du_g}{Dt} + \frac{u_g}{R}\left(-\frac{1}{fR}\frac{\partial M^*}{\partial\Phi}\right) = \frac{\partial M}{\gamma\partial r} + \frac{u_g^2}{R},$$

so that

$$\frac{Du_g}{Dt} - \left(\frac{R}{r}\right) fv + \frac{\partial M}{\gamma \partial r} = 0.$$

With the definition of γ, the above expression can be written

$$\frac{Du_g}{Dt} - \left(f + \frac{v_g}{r}\right) \frac{v}{\gamma} + \frac{\partial M}{\gamma \partial r} = 0, \tag{A.3}$$

which is exactly the momentum equation (2.37) in physical space.

Similarly, from (A.2) we have

$$\frac{D}{Dt}\left(\tfrac{1}{2} f R^2\right) + \frac{\partial M^*}{\partial \Phi} = 0,$$

or,

$$\frac{D}{Dt}\left(\tfrac{1}{2} f r^2 + r v_g\right) + \frac{\partial M}{\partial \phi} = 0$$

by using the potential radius formula (2.55) and the second entry of (2.61). Calculating the derivative in the last expression, we obtain

$$v_g \frac{Dr}{Dt} + r \frac{Dv_g}{Dt} + fr \frac{Dr}{Dt} + \frac{\partial M}{\partial \phi} = 0,$$

or

$$\frac{Dv_g}{Dt} + \left(f + \frac{v_g}{r}\right) u + \frac{\partial M}{r \partial \phi} = 0. \tag{A.4}$$

Collecting (A.3) and (A.4), one should be convinced that their transformed counterparts are (2.64) and (2.65) in $(R, \Phi, S, \mathcal{T})$ space.

A.2 The mixed-balance momentum equations on a sphere

Following the same procedure as in the f-plane case, we start with the transformed canonical equations [Eqs. (3.59) and (3.60) of Chapter 3]:

$$2\Omega \sin \Phi a \frac{D\Phi}{Dt} = \frac{\partial M^*}{a \cos \Phi \partial \Lambda}, \tag{A.5}$$

$$-2\Omega \sin \Phi a \cos \Phi \frac{D\Lambda}{Dt} = \frac{\partial M^*}{a \partial \Phi}. \tag{A.6}$$

On substituting the potential latitude formula (3.49) and the first entry of (3.56) into (A.5), we obtain

$$\frac{D}{Dt}\left(\Omega a \sin^2 \phi - u_g \cos \phi\right) - \frac{\partial M}{a \partial \lambda} = 0.$$

Calculating the derivatives and rearranging terms, we can write the last expression as

$$\frac{Du_g}{Dt} - \left(2\Omega \sin\phi + \frac{u_g \tan\phi}{a}\right) v + \frac{\partial M}{a\cos\phi\partial\lambda} = 0, \tag{A.7}$$

which is exactly the momentum equation (3.31) in physical space.

Similarly, from (A.6) we have

$$\frac{D}{Dt}\left(\lambda + \frac{v_g}{2\Omega a \sin\Phi\cos\Phi}\right) + \frac{1}{2\Omega\sin\Phi a\cos\Phi}\frac{\partial M}{\gamma a\partial\phi} + \frac{v_g^2(\cos^2\Phi - \sin^2\Phi)}{a\sin\Phi\cos\Phi} = 0,$$

by substitutions of the second entry of (3.56) and the geostrophic longitude coordinate (3.51). Calculating the derivatives in this expression gives

$$\frac{u}{a\cos\phi} + \frac{1}{2\Omega a\sin\Phi\cos\Phi}\frac{Dv_g}{Dt} - \frac{v_g(\cos^2\Phi - \sin^2\Phi)}{2\Omega a\sin^2\Phi\cos^2\Phi}\frac{D\Phi}{Dt} + \frac{1}{2\Omega\sin\Phi a\cos\Phi}\frac{\partial M}{\gamma a\partial\phi}$$
$$+\frac{v_g^2(\cos^2\Phi - \sin^2\Phi)}{a\sin\Phi\cos\Phi} = 0,$$

in which the third and the fifth terms cancel when (A.5) and (3.56) are used. After the cancellation of these terms, this equation becomes

$$\frac{Dv_g}{Dt} + \left(\frac{\cos\Phi}{\cos\phi}\right) 2\Omega\sin\Phi u + \frac{\partial M}{\gamma a\partial\phi} = 0.$$

Finally, with the definition of the factor γ in (3.39), we obtain

$$\frac{Dv_g}{Dt} + \left(2\Omega\sin\phi + \frac{u_g\tan\phi}{a}\right)\frac{u}{\gamma} + \frac{\partial M}{\gamma a\partial\phi} = 0. \tag{A.8}$$

Thus, we have proved that the momentum equations (3.31) and (3.32) can be transformed to their canonical forms, which are expressed in (3.59) and (3.60).

Appendix B

DERIVATION OF THE VORTICITY EQUATION FOR THE MIXED-BALANCE THEORY

The mixed-balance theories developed in Chapters 2 and 3 possess the three dimensional vorticity equations, (2.66) and (3.61), which have been expressed as isentropic forms in accordance with the coordinate system we adopted. The detailed derivations of these vorticity equations are given in this appendix. In the first section, we derive the vorticity equation for the f-plane theory, and the result is generalized to the spherical case in the second section.

B.1 The vorticity equation associated with the mixed-balance theory on an f-plane

Although the vorticity equation can be derived from the horizontal momentum equations (2.37) and (2.38), the easist way is to work with the canonical equations. We begin with the transformed momentum equations (2.64) and (2.65):

$$fR\frac{D\Phi}{Dt} = \frac{\partial M^*}{\partial R}, \tag{B.1}$$

$$-f\frac{DR}{Dt} = \frac{\partial M^*}{R\partial\Phi}. \tag{B.2}$$

To derive the vorticity equation, we take the cross derivatives of (B.1) and (B.2) in the form:

$$\underset{(1)}{\frac{\partial(\frac{1}{2}R^2)}{r\partial r}\frac{\partial[(\mathrm{B.1})/R]}{\partial\phi}} - \underset{(2)}{\frac{\partial\Phi}{\partial\phi}\frac{\partial[(\mathrm{B.2})R]}{r\partial r}} - \underset{(3)}{\frac{\partial(\frac{1}{2}R^2)}{\partial\phi}\frac{\partial[(\mathrm{B.1})/R]}{r\partial r}} + \underset{(4)}{\frac{\partial\Phi}{r\partial r}\frac{\partial[(\mathrm{B.2})R]}{\partial\phi}}. \tag{B.12}$$

Let us denote the four terms in this expression as (1), (2), (3) and (4) respectively, and calculate each of these terms separately as follows.

On the right hand side of (B.1) and (B.2):

$$
\begin{aligned}
(1) &= \frac{\partial(\frac{1}{2}R^2)}{r\partial r} f \frac{\partial}{\partial\phi}\left(\frac{D\Phi}{Dt}\right) \\
&= f\frac{\partial(\frac{1}{2}R^2)}{r\partial r}\frac{D}{Dt}\left(\frac{\partial\Phi}{\partial\phi}\right) + f\frac{\partial(\frac{1}{2}R^2)}{r\partial r}\left(\frac{\partial u}{\partial\phi}\frac{\partial\Phi}{\partial r} + \frac{\partial v}{\partial\phi}\frac{\partial\Phi}{r\partial\phi} + \frac{\partial\dot{s}}{\partial s}\frac{\partial\Phi}{\partial s}\right) \\
&= \frac{D}{Dt}\left[f\frac{\partial(\frac{1}{2}R^2)}{r\partial r}\frac{\partial\Phi}{\partial\phi}\right] - f\frac{\partial\Phi}{\partial\phi}\frac{D}{Dt}\left[\frac{\partial(\frac{1}{2}R^2)}{r\partial r}\right] + f\frac{\partial(\frac{1}{2}R^2)}{r\partial r}\left(\frac{\partial u}{\partial\phi}\frac{\partial\Phi}{\partial r}\right. \\
&\quad \left. + \frac{\partial v}{\partial\phi}\frac{\partial\Phi}{r\partial\phi} + \frac{\partial\dot{s}}{\partial\phi}\frac{\partial\Phi}{\partial s}\right);
\end{aligned}
$$

$$
\begin{aligned}
(2) &= \frac{\partial\Phi}{\partial\phi} f \frac{\partial}{r\partial r}\left(R\frac{DR}{Dt}\right) \\
&= f\frac{\partial\Phi}{\partial\phi}\frac{D}{Dt}\left[\frac{\partial(\frac{1}{2}R^2)}{r\partial r}\right] + f\frac{\partial\Phi}{\partial\phi}\left(\frac{\partial(ru)}{r\partial r}\frac{\partial(\frac{1}{2}R^2)}{r\partial r} + \frac{\partial(v/r)}{r\partial r}\frac{\partial(\frac{1}{2}R^2)}{r\partial\phi}\right. \\
&\quad \left. + \frac{\partial\dot{s}}{r\partial r}\frac{\partial(\frac{1}{2}R^2)}{\partial s}\right);
\end{aligned}
$$

$$
\begin{aligned}
(3) &= -\frac{\partial(\frac{1}{2}R^2)}{\partial\phi} f \frac{\partial}{r\partial r}\left(\frac{D\Phi}{Dt}\right) \\
&= -\frac{D}{Dt}\left[f\frac{\partial(\frac{1}{2}R^2)}{\partial\phi}\frac{\partial\Phi}{r\partial r}\right] + f\frac{\partial\Phi}{r\partial r}\frac{D}{Dt}\left[\frac{\partial(\frac{1}{2}R^2)}{\partial\phi}\right] - f\frac{\partial(\frac{1}{2}R^2)}{\partial\phi}\left(\frac{\partial(ru)}{r\partial r}\frac{\partial\Phi}{r\partial r}\right. \\
&\quad \left. + \frac{\partial(v/r)}{r\partial r}\frac{\partial\Phi}{\partial\phi} + \frac{\partial\dot{s}}{r\partial r}\frac{\partial\Phi}{\partial s}\right);
\end{aligned}
$$

$$
\begin{aligned}
(4) &= -\frac{\partial\Phi}{r\partial r} f \frac{\partial}{r\partial\phi}\left(R\frac{DR}{Dt}\right) \\
&= -f\frac{\partial\Phi}{r\partial r}\frac{D}{Dt}\left[\frac{\partial(\frac{1}{2}R^2)}{\partial\phi}\right] - f\frac{\partial\Phi}{r\partial r}\left(\frac{\partial u}{\partial\phi}\frac{\partial(\frac{1}{2}R^2)}{\partial r} + \frac{\partial v}{\partial\phi}\frac{\partial(\frac{1}{2}R^2)}{r\partial\phi} + \frac{\partial\dot{s}}{\partial\phi}\frac{\partial(\frac{1}{2}R^2)}{\partial s}\right).
\end{aligned}
$$

Noting the cancellations among several terms, we now add the four terms together and group them in such a form:

$$
\begin{aligned}
\text{LHS} = \frac{D}{Dt}\left[f\frac{\partial(\frac{1}{2}R^2,\Phi)}{\partial(\frac{1}{2}r^2,\phi)}\right] + f\frac{\partial(\frac{1}{2}R^2,\Phi)}{\partial(\frac{1}{2}^2,\phi)}\left(\frac{\partial(ru)}{r\partial r} + \frac{\partial v}{r\partial\phi}\right) - \left[f\frac{\partial(\frac{1}{2}R^2,\Phi)}{r\partial(\phi,s)}\frac{\partial}{\partial r}\right. \\
\left. + f\frac{\partial(\frac{1}{2}R^2,\Phi)}{r\partial(s,r)}\frac{\partial}{r\partial\phi}\right]\dot{s}. \qquad \text{(B.4)}
\end{aligned}
$$

If we define the vorticity vector as:

$$
(\xi,\eta,\zeta) = f\left(\frac{\partial(\frac{1}{2}R^2,\Phi)}{r\partial(\phi,s)}, \frac{\partial(\frac{1}{2}R^2,\Phi)}{r\partial(s,r)}, \frac{\partial(\frac{1}{2}R^2,\Phi)}{\partial(\frac{1}{2}r^2,\phi)}\right), \qquad \text{(B.5)}
$$

then (B.4) simply becomes

$$\text{LHS} = \frac{D\zeta}{Dt} + \zeta\left(\frac{\partial(ru)}{r\partial r} + \frac{\partial v}{r\partial\phi}\right) - \left(\xi\frac{\partial}{\partial r} + \eta\frac{\partial}{r\partial\phi}\right)\dot{s}. \tag{B.6}$$

We now calculate (B.3) for the right hand side of (B.1) and (B.2) as follows:

$$(1) \;=\; \frac{\partial(\frac{1}{2}R^2)}{r\partial r}\frac{\partial}{\partial\phi}\left(\frac{\partial M^*}{R\partial R}\right);$$

$$\begin{aligned}(2) \;&=\; -\frac{\partial\Phi}{\partial\phi}\frac{\partial}{r\partial r}\left(\frac{\partial M^*}{\partial\Phi}\right) = -\frac{\partial}{r\partial r}\left(\frac{\partial\Phi}{\partial\phi}\frac{\partial M^*}{\partial\Phi}\right) + \frac{\partial M^*}{\partial\Phi}\frac{\partial}{r\partial r}\left(\frac{\partial\Phi}{\partial\phi}\right)\\ &=\; -\frac{\partial}{r\partial r}\left(\frac{\partial\Phi}{\partial\phi}\frac{\partial M^*}{\partial\Phi}\right) + \frac{\partial}{\partial\phi}\left(\frac{\partial\Phi}{r\partial r}\frac{\partial M^*}{\partial\Phi}\right) - \frac{\partial\Phi}{r\partial r}\frac{\partial}{\partial\phi}\left(\frac{\partial M^*}{\partial\Phi}\right);\end{aligned}$$

$$\begin{aligned}(3) \;&=\; -\frac{\partial(\frac{1}{2}R^2)}{\partial\phi}\frac{\partial}{r\partial r}\left(\frac{\partial M^*}{R\partial R}\right) = -\frac{\partial}{r\partial r}\left(\frac{\partial(\frac{1}{2}R^2)}{\partial\phi}\frac{\partial M^*}{R\partial R}\right) + \frac{\partial M^*}{R\partial R}\frac{\partial}{r\partial r}\left(\frac{\partial(\frac{1}{2}R^2)}{\partial\phi}\right)\\ &=\; -\frac{\partial}{r\partial r}\left(\frac{\partial(\frac{1}{2}R^2)}{\partial\phi}\frac{\partial M^*}{R\partial R}\right) + \frac{\partial}{\partial\phi}\left(\frac{\partial(\frac{1}{2}R^2)}{r\partial r}\frac{\partial M^*}{R\partial R}\right) - \frac{\partial(\frac{1}{2}R^2)}{r\partial r}\frac{\partial}{\partial\phi}\left(\frac{\partial M^*}{R\partial R}\right);\end{aligned}$$

$$(4) \;=\; \frac{\partial\Phi}{r\partial r}\frac{\partial}{\partial\phi}\left(\frac{\partial M^*}{\partial\Phi}\right).$$

The addition of these four terms results in the cancellations among several terms, and the remaining terms can be written

$$\begin{aligned}\text{RHS} &= -\frac{\partial}{r\partial r}\left(\frac{\partial\Phi}{\partial\phi}\frac{\partial M^*}{\partial\Phi} + \frac{\partial(\frac{1}{2}R^2)}{\partial\phi}\frac{\partial M^*}{R\partial R}\right) + \frac{\partial}{\partial\phi}\left(\frac{\partial\Phi}{r\partial r}\frac{\partial M^*}{\partial\Phi} + \frac{\partial(\frac{1}{2}R^2)}{r\partial r}\frac{\partial M^*}{R\partial R}\right)\\ &= -\frac{\partial}{r\partial r}\left(\frac{\partial M^*}{\partial\phi}\right) + \frac{\partial}{\partial\phi}\left(\frac{\partial M^*}{r\partial r}\right) = 0.\end{aligned} \tag{B.7}$$

The last line is obtained by using the rules of differentiation or the derivative relations for the coordinate transformation given in (2.58) and (2.59) of Chapter 2.

Now collecting results (B.5), (B.6) and (B.7), we have derived the vorticity equation for the mixed balanced theory on an f-plane:

$$\frac{D\zeta}{Dt} + \zeta\left(\frac{\partial(ru)}{r\partial r} + \frac{\partial v}{r\partial\phi}\right) - \left(\xi\frac{\partial}{\partial r} + \eta\frac{\partial}{r\partial\phi}\right)\dot{s} = 0. \tag{B.8}$$

where (ξ, η, ζ) is the three dimensional vorticity defined in (B.5).

B.2 The vorticity equation associated with the mixed-balance theory on a sphere

We begin with the transformed momentum equations (3.59) and (3.60):

$$2\Omega\sin\Phi a\frac{D\Phi}{Dt} = \frac{\partial M^*}{a\cos\Phi\partial\Lambda}, \tag{B.9}$$

$$-2\Omega\sin\Phi a\cos\Phi\frac{D\Lambda}{Dt} = \frac{\partial M^*}{a\partial\Phi}. \tag{B.10}$$

To derive the vorticity equation, we take the cross derivatives of (B.9) and (B.10) in the form:

$$\underbrace{\frac{\partial\Lambda}{\partial\lambda}\frac{\partial[(\mathrm{B.9})\cos\Phi]}{\partial\sin\phi}}_{(1)}-\underbrace{\frac{\partial\sin\Phi}{\partial\sin\phi}\frac{\partial[(\mathrm{B.10})/\cos\Phi]}{\partial\lambda}}_{(2)}-\underbrace{\frac{\partial\Lambda}{\partial\sin\phi}\frac{\partial[(\mathrm{B.9})\cos\Phi]}{\partial\lambda}}_{(3)}+\underbrace{\frac{\partial\sin\Phi}{\partial\lambda}\frac{\partial[(\mathrm{B.10})/\cos\Phi]}{\partial\sin\phi}}_{(4)}. \tag{B.11}$$

Let us denote the four terms in this expression as (1), (2), (3) and (4) respectively, and calculate each of these terms separately as follows.

On the right hand side of (B.9) and (B.10):

$$\begin{aligned}
(1) &= \frac{\partial\Lambda}{\partial\lambda}\frac{\partial}{\partial\sin\phi}\left(2\Omega\sin\Phi a\frac{D(\sin\Phi)}{Dt}\right)\\
&= \frac{\partial\Lambda}{\partial\lambda}\frac{D(\sin\Phi)}{Dt}\frac{\partial(2\Omega a\sin\Phi)}{\partial\sin\phi}+\frac{\partial\Lambda}{\partial\lambda}2\Omega a\sin\Phi\frac{\partial}{\partial\sin\phi}\left(\frac{D(\sin\Phi)}{Dt}\right)\\
&= \frac{\partial\Lambda}{\partial\lambda}\frac{D(\sin\Phi)}{Dt}\frac{\partial(2\Omega a\sin\Phi)}{\partial\sin\phi}+2\Omega a\sin\Phi\frac{D}{Dt}\left(\frac{\partial\Lambda}{\partial\lambda}\frac{\partial(\sin\Phi)}{\partial\sin\phi}\right)\\
&\quad -2\Omega a\sin\Phi\frac{\partial(\sin\Phi)}{\partial\sin\phi}\frac{D}{Dt}\left(\frac{\partial\Lambda}{\partial\lambda}\right)+2\Omega a\sin\Phi\frac{\partial\Lambda}{\partial\lambda}\left[u\frac{\partial(\sin\Phi)}{a\cos\phi\partial\lambda}\frac{\partial}{\partial\phi}\left(\frac{1}{\cos\phi}\right)\right.\\
&\quad \left.+\frac{\partial u}{\partial\sin\phi}\frac{\partial(\sin\Phi)}{a\cos\phi\partial\lambda}+\frac{\partial(\sin\Phi)}{\partial\sin\phi}\frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi}+\frac{\partial\dot{s}}{\partial\sin\phi}\frac{\partial(\sin\Phi)}{\partial s}\right]
\end{aligned}$$

$$\begin{aligned}
(2) &= \frac{\partial(\sin\Phi)}{\partial\sin\phi}\frac{\partial}{\partial\lambda}\left(2\Omega a\sin\Phi\frac{D\Lambda}{Dt}\right)\\
&= \frac{\partial(\sin\Phi)}{\partial\sin\phi}\frac{D\Lambda}{Dt}\frac{\partial(2\Omega a\sin\Phi)}{\partial\lambda}+2\Omega a\sin\Phi\frac{\partial(\sin\Phi)}{\partial\sin\phi}\frac{\partial}{\partial\lambda}\left(\frac{D\Lambda}{Dt}\right)\\
&= \frac{\partial(\sin\Phi)}{\partial\sin\phi}\frac{D\Lambda}{Dt}\frac{\partial(2\Omega a\sin\Phi)}{\partial\lambda}+2\Omega a\sin\Phi\frac{\partial(\sin\Phi)}{\partial\sin\phi}\frac{D}{Dt}\left(\frac{\partial\Lambda}{\partial\lambda}\right)\\
&\quad +2\Omega a\sin\Phi\frac{\partial(\sin\Phi)}{\partial\sin\phi}\left[\frac{\partial u}{\partial\lambda}\frac{\partial\Lambda}{a\cos\phi\partial\lambda}+\frac{\partial v}{\partial\lambda}\frac{\partial\Lambda}{a\partial\phi}+\frac{\partial\dot{s}}{\partial\lambda}\frac{\partial\Lambda}{\partial s}\right]
\end{aligned}$$

$$
\begin{aligned}
(3) \quad = \quad & -\frac{\partial \Lambda}{\partial \sin\phi}\frac{\partial}{\partial \lambda}\left(2\Omega \sin\Phi a \frac{D(\sin\Phi)}{Dt}\right) \\
= \quad & -\frac{\partial \Lambda}{\partial \sin\phi}\frac{D(\sin\Phi)}{Dt}\frac{\partial(2\Omega a \sin\Phi)}{\partial \lambda} - 2\Omega \sin\Phi \frac{\partial \Lambda}{\partial \sin\phi}\frac{\partial}{\partial \lambda}\left(\frac{D(\sin\Phi)}{Dt}\right) \\
= \quad & -\frac{\partial \Lambda}{\partial \sin\phi}\frac{D(\sin\Phi)}{Dt}\frac{\partial(2\Omega a \sin\Phi)}{\partial \lambda} - 2\Omega \sin\Phi \frac{D}{Dt}\left(\frac{\partial \Lambda}{\partial \sin\phi}\frac{\partial(\sin\Phi)}{\partial \lambda}\right) \\
& +2\Omega \sin\Phi \frac{\partial(\sin\Phi)}{\partial \lambda}\frac{D}{Dt}\left(\frac{\partial \Lambda}{\partial \sin\phi}\right) \\
& -2\Omega \sin\Phi \frac{\partial \Lambda}{\partial \sin\phi}\left[\frac{\partial u}{\partial \lambda}\frac{\partial(\sin\Phi)}{a\cos\phi\partial\lambda} + \frac{\partial v}{\partial \lambda}\frac{\partial(\sin\Phi)}{a\partial\phi} + \frac{\partial \dot{s}}{\partial \lambda}\frac{\partial(\sin\Phi)}{\partial s}\right]
\end{aligned}
$$

$$
\begin{aligned}
(4) \quad = \quad & -\frac{\partial(\sin\Phi)}{\partial \lambda}\frac{\partial}{\partial \sin\phi}\left(2\Omega a \sin\Phi \frac{D\Lambda}{Dt}\right) \\
= \quad & -\frac{\partial(\sin\Phi)}{\partial \lambda}\frac{D\Lambda}{Dt}\frac{\partial(2\Omega a \sin\Phi)}{\partial \sin\phi} - 2\Omega a \sin\Phi \frac{\partial(\sin\Phi)}{\partial \lambda}\frac{\partial}{\partial \sin\phi}\left(\frac{D\Lambda}{Dt}\right) \\
= \quad & -\frac{\partial(\sin\Phi)}{\partial \lambda}\frac{D\Lambda}{Dt}\frac{\partial(2\Omega a \sin\Phi)}{\partial \sin\phi} - 2\Omega a \sin\Phi \frac{\partial(\sin\Phi)}{\partial \lambda}\frac{D}{Dt}\left(\frac{\partial \Lambda}{\partial \sin\phi}\right) \\
& -2\Omega a \sin\Phi \frac{\partial(\sin\Phi)}{\partial \lambda}\left[u\frac{\partial \Lambda}{a\cos\phi\partial\lambda}\frac{\partial}{\partial\phi}\left(\frac{1}{\cos\phi}\right) + \frac{\partial u}{\partial \sin\phi}\frac{\partial \Lambda}{a\cos\phi\partial\lambda}\right. \\
& \left. +\frac{\partial \Lambda}{\partial \sin\phi}\frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi} + \frac{\partial \dot{s}}{\partial \sin\phi}\frac{\partial(\sin\Phi)}{\partial s}\right]
\end{aligned}
$$

Noting the cancellations among several terms, we now add the four terms together and group them in such a form:

$$
\begin{aligned}
\text{LHS} = & \frac{D}{Dt}\left[2\Omega \sin\Phi \frac{\partial(\Lambda, \sin\Phi)}{\partial(\lambda, \sin\phi)}\right] + 2\Omega \sin\Phi \frac{\partial(\Lambda, \sin\Phi)}{\partial(\lambda, \sin\phi)}\left[\frac{\partial u}{a\cos\phi\partial\lambda} + \frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi}\right] \\
& -\left[2\Omega \sin\Phi \frac{\partial(\Lambda, \sin\Phi)}{\partial(\sin\phi, s)}\frac{\partial}{\partial \lambda} + 2\Omega \sin\Phi \frac{\partial(\Lambda, \sin\Phi)}{\partial(s, \lambda)}\frac{\partial}{\partial \sin\phi}\right]\dot{s}. \qquad \text{(B.12)}
\end{aligned}
$$

If we define the vorticity vector as:

$$
(\xi, \eta, \zeta) = 2\Omega \sin\Phi \left(\frac{\partial(\Lambda, \sin\Phi)}{\partial(\sin\phi, s)}, \frac{\partial(\Lambda, \sin\Phi)}{\partial(s, \lambda)}, \frac{\partial(\Lambda, \sin\Phi)}{\partial(\lambda, \sin\phi)}\right), \qquad \text{(B.13)}
$$

then (B.12) simply becomes

$$
\text{LHS} = \frac{D\zeta}{Dt} + \zeta\left(\frac{\partial u}{a\cos\phi\partial\lambda} + \frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi}\right) - \left(\xi\frac{\partial}{\partial \lambda} + \eta\frac{\partial}{\partial \sin\phi}\right)\dot{s}. \qquad \text{(B.14)}
$$

We now calculate (B.11) for the right hand side of (B.9) and (B.10) as follows:

$$(1) \quad = \quad \frac{\partial \Lambda}{\partial \lambda}\frac{\partial}{\partial \sin\phi}\left(\frac{\partial M^*}{\partial \Lambda}\right);$$

$$\begin{aligned}(2) \quad &= \quad -\frac{\partial(\sin\Phi)}{\partial \sin\phi}\frac{\partial}{\partial \lambda}\left(\frac{\partial M^*}{\partial \sin\Phi}\right) = -\frac{\partial}{\partial \lambda}\left(\frac{\partial(\sin\Phi)}{\partial \sin\phi}\frac{\partial M^*}{\partial \sin\Phi}\right) + \frac{\partial M^*}{\partial \sin\Phi}\frac{\partial}{\partial \lambda}\left(\frac{\partial(\sin\Phi)}{\partial \sin\phi}\right) \\ &= \quad -\frac{\partial}{\partial \lambda}\left(\frac{\partial(\sin\Phi)}{\partial \sin\phi}\frac{\partial M^*}{\partial \sin\Phi}\right) + \frac{\partial}{\partial \sin\phi}\left(\frac{\partial(\sin\Phi)}{\partial \lambda}\frac{\partial M^*}{\partial \Phi}\right) - \frac{\partial(\sin\Phi)}{\partial \lambda}\frac{\partial}{\partial \sin\phi}\left(\frac{\partial M^*}{\partial \sin\Phi}\right);\end{aligned}$$

$$\begin{aligned}(3) \quad &= \quad -\frac{\partial \Lambda}{\partial \sin\phi}\frac{\partial}{\partial \lambda}\left(\frac{\partial M^*}{\partial \Lambda}\right) = -\frac{\partial}{\partial \lambda}\left(\frac{\partial \Lambda}{\partial \sin\phi}\frac{\partial M^*}{\partial \Lambda}\right) + \frac{\partial M^*}{\partial \Lambda}\frac{\partial}{\partial \lambda}\left(\frac{\partial \Lambda}{\partial \sin\phi}\right) \\ &= \quad -\frac{\partial}{\partial \lambda}\left(\frac{\partial \Lambda}{\partial \sin\phi}\frac{\partial M^*}{\partial \Lambda}\right) + \frac{\partial}{\partial \sin\phi}\left(\frac{\partial \Lambda}{\partial \lambda}\frac{\partial M^*}{\partial \Lambda}\right) - \frac{\partial \Lambda}{\partial \lambda}\frac{\partial}{\partial \sin\phi}\left(\frac{\partial M^*}{\partial \Lambda}\right);\end{aligned}$$

$$(4) \quad = \quad \frac{\partial(\sin\Phi)}{\partial \lambda}\frac{\partial}{\partial \sin\phi}\left(\frac{\partial M^*}{\partial \sin\Phi}\right).$$

The addition of these four terms results in cancellation among several terms, and the remaining terms can be written

$$\begin{aligned}\text{RHS} = -\frac{\partial}{\partial \lambda}\left(\frac{\partial(\sin\Phi)}{\partial \phi}\frac{\partial M^*}{\partial \Phi} + \frac{\partial \Lambda}{\partial \sin\phi}\frac{\partial M^*}{\partial \Lambda}\right) + \frac{\partial}{\partial \sin\phi}\left(\frac{\partial(\sin\Phi)}{\partial \lambda}\frac{\partial M^*}{\partial \sin\Phi} + \frac{\partial \Lambda}{\partial \lambda}\frac{\partial M^*}{\partial \Lambda}\right) \\ = -\frac{\partial}{\partial \lambda}\left(\frac{\partial M^*}{\partial \sin\phi}\right) + \frac{\partial}{\partial \sin\phi}\left(\frac{\partial M^*}{\partial \lambda}\right) = 0. \qquad \text{(B.15)}\end{aligned}$$

Now collecting results (B.13), (B.14) and (B.15), we have derived the vorticity equation for the mixed-balance theory on the sphere:

$$\frac{D\zeta}{Dt} + \zeta\left(\frac{\partial u}{a\cos\phi\partial\lambda} + \frac{\partial(v\cos\phi)}{a\cos\phi\partial\phi}\right) - \left(\xi\frac{\partial}{\partial\lambda} + \eta\frac{\partial}{\partial\sin\phi}\right)\dot{s} = 0, \qquad \text{(B.16)}$$

where (ξ, η, ζ) is the vector vorticity defined in (B.13).